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Jean-Yves Charbonnel

ON A VARIETY RELATED TO THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA.

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Abstract. For a reductive Lie algbera over an algbraically closed field of charasteristic zero, we consider a Borel subgroup B of its adjoint group, a Cartan subalgebra contained in the Lie algebra of B and the closure X of its orbit under B in the Grassmannian. The variety X plays an important role in the study of the commuting variety. In this note, we prove that X is Gorenstein with rational singularities.

Contents

1. Introduction

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. **Main results.** Let X be the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of B. By a well known result, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of G. By [Ri79], the commuting variety of $\mathfrak{g}$ is the image by the canonical projection of the restriction to $G.X$ of the canonical vector bundle of rank 2ℓ over $\mathrm{Gr}_\ell(\mathfrak{g})$. So X and $G.X$ play an important role in the study of the commuting variety. As it is explained in [CZ16], X and $G.X$ play the same role for the so called generalized commuting varieties and the so called generalized isospectral commuting varieties. The main result of this note is the following theorem:

Theorem 1.1. *The variety X is Gorenstein with rational singulatrities.*

An induction is used to prove this theorem. So we introduce the categories $\mathcal{C}'_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}$ with $\mathfrak{t}$ a commutative Lie algebra of finite dimension. Their objects are nilpotent Lie algebras of finite dimension, normalized by $\mathfrak{t}$ with additional conditions analogous to those of the action of $\mathfrak{h}$ in $\mathfrak{u}$. In particular the minimal dimension

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of the objects in $\mathcal{C}_{\mathfrak{t}}$ is the dimension of $\mathfrak{t}$ and an object of dimension $\dim \mathfrak{t}$ is a commutative algebra. The category $\mathcal{C}_{\mathfrak{t}}$ is a full subcategory of $\mathcal{C}'_{\mathfrak{t}}$. For $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$, we consider the solvable Lie algebra $\mathfrak{r} := \mathfrak{t} + \mathfrak{a}$ and R the adjoint group of $\mathfrak{r}$. Denoting by X_R the closure in $\mathrm{Gr}_{\dim \mathfrak{t}}(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R, we prove by induction on $\dim \mathfrak{a}$ the following theorem:

Theorem 1.2. *The variety X_R is normal and Cohen-Macaulay.*

The result for the category $\mathcal{C}'_{\mathfrak{t}}$ is easily deduced from the result for the category $\mathcal{C}_{\mathfrak{t}}$ by Corollary 2.2. One of the key argument in the proof is the consideration of the fixed points under the action of R in X_R. As a matter of fact, since the closure of all orbit under R in X_R contains a fixed point, X_R is Cohen-Macaulay if so are the fixed points by openness of the set of Cohen-Macaulay points. Then, by Serre's normality criterion, it suffices to prove that X_R is smooth in codimension 1. For that purpose the consideration of the restriction to X_R of the tautological vector bundle of rank $\dim \mathfrak{t}$ over $\mathrm{Gr}_{\dim \mathfrak{t}}(\mathfrak{r})$ is very useful.

For the study of the fixed points, we introduce Property (**P**) and Property (**P**$_1$) for the objects of $\mathcal{C}'_{\mathfrak{t}}$:

- Property (**P**) for $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ says that for V in X_R, contained in the centralizer $\mathfrak{r}^s$ of an element s of $\mathfrak{t}$, V is in the closure of the orbit of $\mathfrak{t}$ under the centralizer R^s of s in R,
- Property (**P**$_1$) for $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ says that for V in X_R normalized by $\mathfrak{t}$ and such that $V \cap \mathfrak{t}$ is the center of $\mathfrak{r}$, then the non zero weights of $\mathfrak{t}$ in V are linearly independent.

Property (**P**$_1$) for $\mathfrak{a}$ results from Property (**P**) for $\mathfrak{a}$ and Property (**P**) for $\mathfrak{a}$ results from Property (**P**$_1$) for $\mathfrak{a}$ and Property (**P**) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ of dimension smaller than $\dim \mathfrak{a}$. So, the main result for the objects of $\mathcal{C}'_{\mathfrak{t}}$ is the following proposition:

Proposition 1.3. *The objects of $\mathcal{C}'_{\mathfrak{t}}$ have Property* (**P**).

From this proposition, we deduce some structure property for the points of X_R.

The second part of Theorem 1.1, that is Gorensteinness property and Rational singularities, is obtained by considering a subcategory $\mathcal{C}_{\mathfrak{t},*}$ of $\mathcal{C}_{\mathfrak{t}}$. This category is defined by an additional condition on the objects. The main point for $\mathfrak{a}$ in $\mathcal{C}_{\mathfrak{t},*}$ is the following result:

Proposition 1.4. *Let $k \geqslant 2$ be an integer. Denote by $\mathcal{E}^{(k)}$ the R-equivariant vector subbundle of $X_R \times \mathfrak{r}^k$ whose fiber at $\mathfrak{t}$ is $\mathfrak{t}^k$. Then there exists on the smooth locus of $\mathcal{E}^{(k)}$ a regular differential form of top degree without zero.*

From Proposition 1.4 and Theorem 1.2, we deduce that $\mathcal{E}^{(k)}$ and X_R are Gorenstein with rational singularities.

This note is organized as follows. In Section 2, categories $\mathcal{C}'_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}$ are introduced for some space $\mathfrak{t}$. In particular, $\mathfrak{u}$ is an object of $\mathcal{C}_{\mathfrak{h}}$. In Subsection 2.3, we define Property (**P**) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ and we deduce some result on the structure of points of X_R. In Subsection 2.4, we define Property (**P**$_1$) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ and we prove that Property (**P**)$_1$ is a consequence of Property (**P**). In Subsection 2.5, we give some geometric constructions to prove Property (**P**) by induction on the dimension of $\mathfrak{a}$. At last, in Subsection 2.6, we prove Proposition 1.3. In particular, the proof of [CZ16, Lemma 4.4,(i)] is completed. In Section 3, we are interested in the singular locus of X_R. In Subsection 3.3, regularity in codimension 1 is proved with some additional properties analogous to those of [CZ16, Section 3]. Moreover, the constructions of Subsection 2.5 are used to prove the results by induction on the dimension of $\mathfrak{a}$. In Section 4, Cohen-Macaulayness property is proved by induction. In Section 5, the category $\mathcal{C}_{\mathfrak{t},*}$ is introduced and Proposition 1.4 is proved. Then with some results given in the appendix, we finish the proof of Theorem 1.1.

1.2. **Notations.** • An algebraic variety is a reduced scheme over $\Bbbk$ of finite type. For X an algebraic variety, its smooth locus is denoted by $X_{\rm sm}$.

• Set $\Bbbk^* := \Bbbk \setminus \{0\}$. For V a vector space, its dual is denoted by V^*.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\Bbbk[X]$ is the algebra of regular functions on X, $\Bbbk(X)$ is the field of rational functions on X when X is irreducible and Ω_X is the sheaf of regular differential forms of top degree on X when X is smooth and irreducible.

• If E is a subset of a vector space V, denote by span(E) the vector subspace of V generated by E. The grassmannian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$.

• For $\mathfrak{a}$ a Lie algebra,V a subspace of $\mathfrak{a}$ and x in $\mathfrak{a}$, V^x denotes the centralizer of x in V. For A a subgroup of the group of automorphisms of $\mathfrak{a}$, A^x denotes the centralizer of x in A. An element x of $\mathfrak{g}$ is regular if $\mathfrak{g}^x$ has dimension ℓ and the set of regular elements of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\rm reg}$.

• The Lie algebra of an algberaic torus is also called a torus. In this note, a torus denoted by a gothic letter means the Lie algebra of an algebraic torus.

• For $\mathfrak{a}$ a Lie algebra, the Lie algebra of derivations of $\mathfrak{a}$ is denoted by $\mathrm{Der}(\mathfrak{a})$. By definition $\mathrm{Der}(\mathfrak{a})$ is the Lie algebra of the group $\mathrm{Aut}(\mathfrak{a})$ of the automorphisms of $\mathfrak{a}$.

• Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$ and $\mathfrak{u}$ the nilpotent radical of $\mathfrak{b}$.

2. On solvable algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d. Denote by $\tilde{\mathcal{C}}_{\mathfrak{t}}$ the subcategory of the category of finite dimensional Lie algebras whose objects are finite dimensional nilpotent Lie algebras $\mathfrak{a}$ such that there exists a morphism

$$\mathfrak{t} \xrightarrow{\varphi_{\mathfrak{a}}} \mathrm{Der}(\mathfrak{a})$$

whose image is the Lie algebra of a subtorus of $\mathrm{Aut}(\mathfrak{a})$. For $\mathfrak{a}$ and $\mathfrak{a}'$ in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, a morphism ψ from $\mathfrak{a}$ to $\mathfrak{a}'$ is a morphism of Lie algebras such that $\psi\circ\varphi_{\mathfrak{a}}(t) = \varphi_{\mathfrak{a}'}(t)\circ\psi$ for all t in $\mathfrak{t}$. For x in $\mathfrak{t}$, x is a semisimple derivation of $\mathfrak{a}$. Denote by $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$ the set of weights of $\mathfrak{t}$ in $\mathfrak{a}$. Let $\mathcal{C}'_{\mathfrak{t}}$ be the full subcategory of objects $\mathfrak{a}$ of $\tilde{\mathcal{C}}_{\mathfrak{t}}$ verifying the following conditions:

(1) 0 is not in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$,
(2) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, the weight space of weight α has dimension 1,
(3) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\Bbbk\alpha \cap (\mathcal{R}_{\mathfrak{t},\mathfrak{a}} \setminus \{\alpha\})$ is empty.

For $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ and $\mathfrak{a}'$ a subalgebra of $\mathfrak{a}$, invariant under the adjoint action of $\mathfrak{t}$, $\mathfrak{a}'$ is in $\mathcal{C}'_{\mathfrak{t}}$. Denote by $\mathcal{C}_{\mathfrak{t}}$ the full subcategory of objects $\mathfrak{a}$ of $\mathcal{C}'_{\mathfrak{t}}$ such that $\varphi_{\mathfrak{a}}$ is an embedding. For example $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{h}}$.

For $\mathfrak{a}$ in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, denote by $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ the solvable algebra $\mathfrak{t} + \mathfrak{a}$, $\pi_{\mathfrak{t},\mathfrak{a}}$ the quotient morphism from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to $\mathfrak{t}$, $R_{\mathfrak{t},\mathfrak{a}}$ the adjoint group of $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$, $A_{\mathfrak{t},\mathfrak{a}}$ the connected closed subgroup of $R_{\mathfrak{t},\mathfrak{a}}$ whose Lie algebra is $\mathrm{ad}\,\mathfrak{a}$, $X_{R_{\mathfrak{t},\mathfrak{a}}}$ the closure in $\mathrm{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ of the orbit of $\mathfrak{t}$ under $R_{\mathfrak{t},\mathfrak{a}}$ and $\mathcal{E}_{\mathfrak{t},\mathfrak{a}}$ the restriction to $X_{R_{\mathfrak{t},\mathfrak{a}}}$ of the tautological vector bundle over $\mathrm{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$. The variety $X_{R_{\mathfrak{t},\mathfrak{a}}}$ is called *the main variety related to* $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$. For α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, let $\mathfrak{a}^{\alpha}$ be the weight space of weight α under the action of $\mathfrak{t}$ in $\mathfrak{a}$.

In the following subsections, a vector space $\mathfrak{t}$ of positive dimension d and an object $\mathfrak{a}$ of $\mathcal{C}'_{\mathfrak{t}}$ are fixed. We set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

Let $\mathfrak{z}$ be the orthogonal complement of $\mathcal{R}$ in $\mathfrak{t}$ and $d^{\#}$ its codimension in $\mathfrak{t}$. Then $n \geqslant d^{\#}$.

2.1. **General remarks on $\mathcal{C}'_{\mathfrak{t}}$.** For x in $\mathfrak{r}$, we say that x is semisimple if so is $\operatorname{ad} x$ and x is nilpotent if so is $\operatorname{ad} x$. For $\mathfrak{s}$ a commutative subalgebra of $\mathfrak{r}$, we say that $\mathfrak{s}$ is a torus if $\operatorname{ad} \mathfrak{s}$ is the Lie algebra of a subtorus of $\mathrm{GL}(\mathfrak{r})$.

Lemma 2.1. *Let x be in $\mathfrak{r}$ and $\mathfrak{s}$ a commutative subalgebra of $\mathfrak{r}$.*

(i) *The center of $\mathfrak{r}$ is equal to $\mathfrak{z}$.*

(ii) *The element x is semisimple if and only if $R.x \cap \mathfrak{t}$ is not empty.*

(iii) *The element x is nilpotent if and only if x is in $\mathfrak{z} + \mathfrak{a}$.*

(iv) *The algebra $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$ if and only if $\mathfrak{z} = \{0\}$. In this case, x has a unique decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ with $[x_{\mathrm{s}}, x_{\mathrm{n}}] = 0$, x_{s} semisimple and x_{n} nilpotent.*

(v) *The algebra $\mathfrak{s}$ is a torus if and only if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$. In this case, $\mathfrak{s}$ and $\pi(\mathfrak{s})$ are conjugate under R.*

Proof. By definition $\operatorname{ad} \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ is an algebraic solvable subalgebra of $\mathfrak{gl}(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ and $\operatorname{ad} \mathfrak{t}$ is a maximal subtorus of $\operatorname{ad} \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$.

(i) Let $\mathfrak{z}'$ be the center of $\mathfrak{r}$. As $[\mathfrak{t}, \mathfrak{z}'] = \{0\}$,

$$\mathfrak{z}' = \mathfrak{z}' \cap \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{z}' \cap \mathfrak{a}^{\alpha}.$$

So, by Condition (1), $\mathfrak{z}'$ is contained in $\mathfrak{t}$. For t in $\mathfrak{t}$, t is in $\mathfrak{z}'$ if and only if $\alpha(t) = 0$ for all α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, whence $\mathfrak{z}' = \mathfrak{z}$.

(ii) As the elements of $\mathfrak{t}$ are semisimple by defintion, the condition is sufficient since the set of semisimple elements of $\mathfrak{r}$ is invariant under the adjoint action of R. Suppose that x is semisimple. By [Hu95, Ch. VII], for some g in R, $\operatorname{Ad} g(x)$ is in $\operatorname{ad} \mathfrak{t}$, whence $g(x)$ is in $\mathfrak{t}$ by (i).

(iii) As $\operatorname{ad} \mathfrak{a}$ is the set of nilpotent elements of $\operatorname{ad} \mathfrak{r}$, x is in $\mathfrak{z} + \mathfrak{a}$ if and only if it is nilpotent by (i).

(iv) By definition, $\mathfrak{z}$ is the kernel of $\varphi_{\mathfrak{a}}$. Hence $\mathfrak{z} = \{0\}$ if and only if $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$. As $\operatorname{ad} \mathfrak{r}$ is an algebraic subalgebra of $\mathrm{gl}(\mathfrak{r})$, it contains the components of the Jordan decomposition of $\operatorname{ad} x$. As a result, when $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$, x has a unique decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ with $[x_{\mathrm{s}}, x_{\mathrm{n}}] = 0$, x_{s} semisimple and x_{n} nilpotent.

(v) Suppose that $\mathfrak{s}$ is a torus. By (i), $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and by [Hu95, Ch. VII], for some g in R, $\operatorname{ad} g(\mathfrak{s})$ is contained in $\operatorname{ad} \mathfrak{t}$ since $\operatorname{ad} \mathfrak{t}$ is a maximal torus of $\operatorname{ad} \mathfrak{r}$. Then, by (i), $g(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$. Moreover, $g(\mathfrak{s}) = \pi(\mathfrak{s})$ since $g(y) - y$ is in $\mathfrak{a}$ for all y in $\mathfrak{r}$. Conversely, if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$, $\operatorname{ad} \mathfrak{s}$ is conjugate to the subtorus $\operatorname{ad} \pi(\mathfrak{s})$ of $\operatorname{ad} \mathfrak{t}$ by [Hu95, Ch. VII] so that $\mathfrak{s}$ and $\pi(\mathfrak{s})$ are conjugate under R. □

Denoting by $\mathfrak{t}^{\#}$ a complement to $\mathfrak{z}$ in $\mathfrak{t}$, $\mathfrak{a}$ is an object of $\mathcal{C}_{\mathfrak{t}^{\#}}$ since $\varphi_{\mathfrak{a}}(\mathfrak{t}) = \varphi_{\mathfrak{a}}(\mathfrak{t}^{\#})$ and the restriction of $\varphi_{\mathfrak{a}}$ to $\mathfrak{t}^{\#}$ is injective. Set $\mathfrak{r}^{\#} := \mathfrak{t}^{\#} + \mathfrak{a}$ and denote by $R^{\#}$ the adjoint group of $\mathfrak{r}^{\#}$. Let $X_{R^{\#}}$ be the closure in $\mathrm{Gr}_{d^{\#}}(\mathfrak{r}^{\#})$ of the orbit of $\mathfrak{t}^{\#}$ under $R^{\#}$.

Corollary 2.2. *All element of X_R is a commutative algebra containing $\mathfrak{z}$. Moreover, the map*

$$X_{R^{\#}} \longrightarrow X_R\ , \qquad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism.

Proof. As the set of commutative subalgebras of dimension d of $\mathfrak{r}$ is a closed subset of $\mathrm{Gr}_d(\mathfrak{r})$ containing $\mathfrak{t}$ and invariant under R, all element of X_R is a commutative algebra. According to Lemma 2.1(i), all element of $R.\mathfrak{t}$ contains $\mathfrak{z}$ and so does all element of X_R. For g in R, denote by $\overline{g}$ the image of g in $R^{\#}$ by the restriction morphism. Then

$$g(\mathfrak{t}) = \overline{g}(\mathfrak{t}^{\#}) + \mathfrak{z} \quad \text{and} \quad \overline{g}(\mathfrak{t}^{\#}) = g(\mathfrak{t}) \cap \mathfrak{r}^{\#}.$$

Hence the map

$$X_{R^{\#}} \longrightarrow X_R \ , \qquad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism whose inverse is the map $V \mapsto V \cap \mathfrak{r}^{\#}$. □

For $\mathfrak{a}$ of dimension $d^{\#}$, $\mathcal{R} := \{\beta_1, \dots, \beta_{d^{\#}}\}$, and for I subset of $\{1, \dots, d^{\#}\}$, denote $X_{R,I}$ the image of $\Bbbk^I$ by the map

$$\Bbbk^I \longrightarrow X_R \ , \qquad (z_i,\ i \in I) \longmapsto \mathfrak{z} \oplus \mathrm{span}(\{t_i + z_i x_i,\ i \in I\}) \oplus \bigoplus_{i \notin I} \mathfrak{a}^{\beta_i}$$

with x_i in $\mathfrak{a}^{\beta_i}$ for $i = 1, \dots, d^{\#}$ and $t_1, \dots, t_{d^{\#}}$ in $\mathfrak{t}$ such that $\beta_i(t_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant d^{\#}$, with $\delta_{i,j}$ the Kronecker symbol.

Lemma 2.3. *Suppose that $\mathfrak{a}$ has dimension $d^{\#}$. Denote by $\beta_1, \dots, \beta_{d^{\#}}$ the elements of $\mathcal{R}$.*

(i) *The algebra $\mathfrak{a}$ is commutative.*

(ii) *The set X_R is the union of $X_{R,I},\ I \subset \{1, \dots, d^{\#}\}$.*

Proof. (i) As $\mathfrak{z}$ has codimension $d^{\#}$ in $\mathfrak{t}$, $\beta_1, \dots, \beta_{d^{\#}}$ are linearly independent. Hence for $i \neq j$, $\beta_i + \beta_j$ is not in $\mathcal{R}$. As a result, $\mathfrak{a}$ is commutative.

(ii) According to Corollary 2.2, we can suppose $d = d^{\#}$ so that $t_1, \dots, t_d$ is the dual basis of $\beta_1, \dots, \beta_d$. For I subset of $\{1, \dots, d\}$, denote by I' the complement to I in $\{1, \dots, d\}$ and $\mathfrak{z}_{I'}$ the orthogonal complement to $\beta_i,\ i \in I'$ in $\mathfrak{t}$ and set:

$$V_I := \mathfrak{z}_{I'} \oplus \bigoplus_{i \in I'} \mathfrak{a}^{\beta_i}.$$

By (i), for i in I,

$$\exp(z_1 \mathrm{ad}\, x_1 + \dots + z_d \mathrm{ad}\, x_d)(t_i) = t_i - z_i x_i.$$

Hence $X_{R,I}$ is the orbit of V_I under A and its closure in X_R is the union of $X_{R,J},\ J \subset I$. As a result, X_R is the union of $X_{R,I},\ I \subset \{1, \dots, d\}$ since $X_{R,\{1,\dots,d\}}$ is the orbit of $\mathfrak{t}$ under A. □

2.2. **On some subsets of $\mathcal{R}$.** For α in $\mathcal{R}$, let x_α be in $\mathfrak{a}^\alpha \setminus \{0\}$. For Λ subset of $\mathcal{R}$, denote by $\mathfrak{t}_\Lambda$ the intersection of the kernels of its elements and set:

$$\mathfrak{a}_\Lambda := \bigoplus_{\alpha \in \Lambda} \mathfrak{a}^\alpha \quad \text{and} \quad \mathfrak{r}_\Lambda := \mathfrak{t} \oplus \mathfrak{a}_\Lambda.$$

When Λ has only one element α, set $\mathfrak{t}_\alpha := \mathfrak{t}_\Lambda$.

Definition 2.4. Let Λ be a subset of $\mathcal{R}$. We say that Λ is a complete subset of $\mathcal{R}$ if it contains all element of $\mathcal{R}$ whose kernel contains $\mathfrak{t}_\Lambda$

For Λ complete subset of $\mathcal{R}$, $\mathfrak{a}_\Lambda$ is a subalgebra of $\mathfrak{a}$ and $\mathfrak{r}_\Lambda$ is a subalgebra of $\mathfrak{r}$. In particular, $\mathfrak{a}_\Lambda$ is in $\mathcal{C}'_{\mathfrak{t}}$. In this case, denote by R_Λ the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\,\mathfrak{r}_\Lambda$.

Lemma 2.5. *Let Λ be a complete subset of $\mathcal{R}$, strictly contained in $\mathcal{R}$. Then $\mathfrak{a}_\Lambda$ is contained in an ideal $\mathfrak{a}'$ of $\mathfrak{r}$ of dimension* $\dim \mathfrak{a} - 1$ *and contained in $\mathfrak{a}$.*

Proof. As Λ is complete and strictly contained in $\mathcal{R}$, $\mathfrak{a}_\Lambda$ is a subalgebra of $\mathfrak{r}$, strictly contained in $\mathfrak{a}$. Then, by Lie's Theorem, there is a sequence

$$\mathfrak{a}_\Lambda = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_m = \mathfrak{a}$$

of subalgebras of $\mathfrak{r}$ such that $\mathfrak{a}_i$ is an ideal of codimension 1 of $\mathfrak{a}_{i+1}$ for $i = 0, \ldots, m-1$, whence the lemma. □

For s in $\mathfrak{t}$, denote by Λ_s the subset of elements of $\mathcal{R}$ whose kernel contains s.

Lemma 2.6. *Let s be in* $\mathfrak{t}$.
(i) *The centralizer $\mathfrak{r}^s$ of s in $\mathfrak{r}$ is the direct sum of $\mathfrak{t}$ and $\mathfrak{a}_{\Lambda_s}$.*
(ii) *The center of $\mathfrak{r}^s$ is equal to $\mathfrak{t}_{\Lambda_s}$.*

Proof. By definition, Λ_s is a complete subset of $\mathcal{R}$. Let x be in $\mathfrak{r}$. Then x has a unique decomposition

$$x = x_0 + \sum_{\alpha \in \mathcal{R}} c_\alpha x_\alpha$$

with x_0 in $\mathfrak{t}$ and $c_\alpha, \alpha \in \mathcal{R}$ in $\Bbbk$.

(i) Since s is in $\mathfrak{t}$, x is in $\mathfrak{r}^s$ if and only if $c_\alpha = 0$ for $\alpha \in \mathcal{R} \setminus \Lambda_s$, whence the assertion.

(ii) The algebra $\mathfrak{a}_{\Lambda_s}$ is in $\mathcal{C}'_\mathfrak{t}$ and $\mathfrak{t}_{\Lambda_s}$ is the orthogonal complement to Λ_s in $\mathfrak{t}$. So, by (i) and Lemma 2.1(i), $\mathfrak{t}_{\Lambda_s}$ is the center of $\mathfrak{r}^s$. □

2.3. **Property (P) for objects of $\mathcal{C}_\mathfrak{t}$.** Let $\mathbf{T}$ be the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\,\mathfrak{t}$. For s in $\mathfrak{t}$, denote by X_R^s the subset of elements of X_R contained in $\mathfrak{r}^s$ and $\overline{R^s.\mathfrak{t}}$ the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R^s. Then $\overline{R^s.\mathfrak{t}}$ is contained in X_R^s.

Definition 2.7. Say that $\mathfrak{a}$ has Property (**P**) if X_R^s is equal to $\overline{R^s.\mathfrak{t}}$ for all s in $\mathfrak{t}$.

By Corollary 2.2, $\mathfrak{a}$ has Property (**P**) if and only if the object $\mathfrak{a}$ of $\mathcal{C}_{\mathfrak{t}^\#}$ has Property (**P**).

Lemma 2.8. *If $\mathfrak{a}$ has dimension $d^\#$, then $\mathfrak{a}$ has Property* (**P**).

Proof. According to Corollary 2.2, we can suppose $d = d^\#$. Denote by $\beta_1, \ldots, \beta_d$ the elements of $\mathcal{R}$. Then $\beta_1, \ldots, \beta_d$ is a basis of $\mathfrak{t}^*$. Let $t_1, \ldots, t_d$ be the dual basis, s in $\mathfrak{t}$ and V in X_R^s. By Lemma 2.3(ii), for some subset I of $\{1, \ldots, d\}$, V is in $X_{R,I}$. Then for some $(z_i,\ i \in I)$,

$$V = \mathrm{span}(\{t_i + z_i x_i\ i \in I\}) \oplus \bigoplus_{i \in I'} \mathfrak{a}^{\beta_i}$$

with I' the complement to I in $\{1, \ldots, d\}$ and x_i in $\mathfrak{a}^{\beta_i}$ for $i = 1, \ldots, d$. Setting

$$I'' := I' \cup \{i \in I \mid z_i \neq 0\},$$

for i in $\{1, \ldots, d\}$, i is in I'' if and only if $\beta_i(s) = 0$. So, by Lemma 2.5(i),

$$\mathfrak{r}^s = \mathfrak{t} \oplus \bigoplus_{i \in I''} \mathfrak{a}^{\beta_i}.$$

Then by Lemma 2.3(ii), V is in $\overline{R^s.\mathfrak{t}}$. □

By definition, an *algebraic subalgebra* $\mathfrak{k}$ of $\mathfrak{r}$ is the semi-direct product of a torus $\mathfrak{s}$ contained in $\mathfrak{k}$ and $\mathfrak{k} \cap \mathfrak{a}$.

Lemma 2.9. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in X_R, x in V and y in $\mathfrak{r}$ such that* $\mathrm{ad}\,y$ *is the semisimple component of* $\mathrm{ad}\,x$. *Then the center of $\mathfrak{r}^y$ is contained in V.*

Proof. By Corollary 2.2, we can suppose $\mathfrak{a}$ in $\mathcal{C}_{\mathfrak{t}}$ so that y is the semisimple component of x by Lemma 2.1(iv). By Lemma 2.1(ii), for some g in R, $g(y)$ is in $\mathfrak{t}$. Denote by $\mathfrak{z}_{g(y)}$ the center of $\mathfrak{r}^{g(y)}$. By Lemma 2.6(ii), $\mathfrak{z}_{g(y)}$ is contained in $\mathfrak{t}$. As V is a commutative algebra, $g(V)$ is in $X_R^{g(y)}$. So, by Property (**P**), $\mathfrak{z}_{g(y)}$ is contained in $g(V)$ since $\mathfrak{z}_{g(y)}$ is in $k(\mathfrak{t})$ for all k in $R^{g(y)}$, whence the lemma. $\square$

Corollary 2.10. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in X_R. Then V is a commutative algebraic subalgebra of $\mathfrak{r}$ and for some subset Λ of $\mathcal{R}$, the biggest torus contained in V is conjugate to $\mathfrak{t}_\Lambda$ under R.*

Proof. According to Corollary 2.2, V is a commutative subalgebra of $\mathfrak{r}$ and we can suppose $d = d^{\#}$. Let $\mathfrak{s}$ be the set of semisimple elements of V. Then $\mathfrak{s}$ is a subspace of V. By Lemma 2.9, V contains the semisimple components of its elements so that V is the direct sum of $\mathfrak{s}$ and $V\cap\mathfrak{a}$. Let s be in $\mathfrak{s}$ such that the center of $\mathfrak{r}^s$ has maximal dimension. After conjugation by an element of R, we can suppose that s is in $\mathfrak{t}$. By Lemma 2.6(ii), $\mathfrak{t}_{\Lambda_s}$ is the center of $\mathfrak{r}^s$. Hence, by Lemma 2.9, $\mathfrak{t}_{\Lambda_s}$ is contained in $\mathfrak{s}$. Suppose that the inclusion is strict. A contradiction is expected. Let s' be in $\mathfrak{s}\setminus\mathfrak{t}_{\Lambda_s}$. Since V is contained in $\mathfrak{r}^s$, for some g in R^s, $g(s')$ is in $\mathfrak{t}$. Moreover, $g(\mathfrak{s})$ is the set of semisimple elements of $g(V)$ and $\mathfrak{t}_{\Lambda_s}$ is contained in $g(\mathfrak{s})$. Denoting by Λ' the set of elements of Λ_s whose kernel contains $g(s')$, for some z in $\mathbb{k}^*$, Λ' is the set of elements of $\mathcal{R}$ such that $\alpha(s+zg(s'))=0$. By Lemma 2.9, $\mathfrak{t}_{\Lambda'}$ is contained in $g(V)$. So, by minimality of $|\Lambda_s|$, $\Lambda'=\Lambda_s$ and $g(s')$ is in $\mathfrak{t}_{\Lambda_s}$, whence the contradiction since $g(s')$ is in $g(\mathfrak{s})\setminus\mathfrak{t}_{\Lambda_s}$. As a result, $\mathfrak{t}_{\Lambda_s}=\mathfrak{s}$ and $V=\mathfrak{t}_{\Lambda_s}+V\cap\mathfrak{a}$, whence the corollary. $\square$

2.4. **Fixed points in X_R under T and R.** For V subspace of dimension d of $\mathfrak{r}$, denote by $\mathcal{R}_V$ the set of elements β of $\mathcal{R}$ such that $\mathfrak{a}^\beta$ is contained in V, r_V the rank of $\mathcal{R}_V$ and $\mathfrak{z}_V$ its orthogonal complement in $\mathfrak{t}$ so that $\dim\mathfrak{z}_V = d-r_V$. As $\mathrm{Gr}_d(\mathfrak{r})$ and X_R are projective varieties, the actions of $\mathbf{T}$ and R in these varieties have fixed points since $\mathbf{T}$ and R are connected and solvable.

Definition 2.11. We say that $\mathfrak{a}$ has Property ($\mathbf{P}_1$) if for V fixed point under $\mathbf{T}$ in X_R such that $V\cap\mathfrak{t}=\mathfrak{z}$, $r_V=|\mathcal{R}_V|$.

Lemma 2.12. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in* $\mathrm{Gr}_d(\mathfrak{r})$.

(i) *The element V is a fixed point under* $\mathbf{T}$ *in X_R if and only if V is a commutative subalgebra of $\mathfrak{r}$ and*

$$V=\mathfrak{z}_V\oplus\bigoplus_{\beta\in\mathcal{R}_V}\mathfrak{a}^\beta.$$

In this case, $r_V=|\mathcal{R}_V|$.

(ii) *The element V is a fixed point under R in X_R if and only if V is a commutative ideal of $\mathfrak{r}$ and $\mathfrak{z}$ is the orthogonal complement of $\mathcal{R}_V$ in $\mathfrak{t}$. In this case, $r_V=|\mathcal{R}_V|=d^{\#}$.*

Proof. If V is a fixed point under $\mathbf{T}$,

$$V=V\cap\mathfrak{t}\oplus\bigoplus_{\beta\in\mathcal{R}_V}\mathfrak{a}^\beta.$$

(i) Suppose that V is a fixed point under $\mathbf{T}$ in $X_R\setminus\{\mathfrak{t}\}$. Then $\mathcal{R}_V$ is not empty. Let s be an element of $\mathfrak{z}_V$ such that $\beta(s)\neq 0$ if β is not a linear combination of elements of $\mathcal{R}_V$. Then V is contained in $\mathfrak{r}^s$. So, by Property (**P**), V is in $\overline{R^s.\mathfrak{t}}$. By Lemmma 2.6(i), $\mathfrak{z}_V$ is the center of $\mathfrak{r}^s$. Hence $\mathfrak{z}_V$ is contained in V and $\mathfrak{z}_V=V\cap\mathfrak{t}$ since $V\cap\mathfrak{t}$ is contained in $\mathfrak{z}_V$. As a result, $\mathfrak{z}_V$ has dimension $d-|\mathcal{R}_V|$ and $r_V=|\mathcal{R}_V|$.

Conversely, suppose that V is a commutative algebra and

$$V=\mathfrak{z}_V\oplus\bigoplus_{\beta\in\mathcal{R}_V}\mathfrak{a}^\beta.$$

Set:
$$\mathfrak{a}_V := \bigoplus_{\beta\in\mathcal{R}_V} \mathfrak{a}^\beta, \qquad \mathfrak{r}_V := \mathfrak{t}\oplus\mathfrak{a}_V .$$
Then $\mathfrak{a}_V$ is a commutative Lie algebra and $\mathfrak{a}_V$ is in $\mathcal{C}'_{\mathfrak{t}}$. Moreover, $\mathfrak{z}_V$ is the center of $\mathfrak{r}_V$ by Lemma 2.1(i). By Lemma 2.3(ii), V is in the closure of the orbit of $\mathfrak{t}$ under the action of the adjoint group of $\mathfrak{r}_V$ in $\mathrm{Gr}_d(\mathfrak{r}_V)$, whence the assertion.

(ii) The element V of $\mathrm{Gr}_d(\mathfrak{r})$ is a fixed point under R if and only if V is an ideal of $\mathfrak{r}$. So, by (i), the condition is sufficient. Suppose that V is a fixed point under the action of R in X_R. By (i),
$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta\in\mathcal{R}_V} \mathfrak{a}^\beta .$$
As V is an ideal of $\mathfrak{r}$, $\mathfrak{z}_V$ is contained in the kernel of all elements of $\mathcal{R}$ so that $\mathfrak{z}_V = \mathfrak{z}$. In particular, $|\mathcal{R}_V| = d^\#$ and the elements of $\mathcal{R}_V$ are linearly independent. □

2.5. **On some varieties related to X_R.** Let $\mathfrak{a}'$ be an ideal of $\mathfrak{r}$ of dimension $\dim\mathfrak{a}-1$ and contained in $\mathfrak{a}$. As a subalgebra of $\mathfrak{a}$ normalized by $\mathfrak{t}$, $\mathfrak{a}'$ is in $\mathcal{C}'_{\mathfrak{t}}$. Denote by $\mathfrak{r}'$ the subalgebra $\mathfrak{t}+\mathfrak{a}'$ of $\mathfrak{r}$, A' and R' the connected closed subgroups of R whose Lie algebras are $\mathrm{ad}\,\mathfrak{a}'$ and $\mathrm{ad}\,\mathfrak{r}'$ respectively. Let $X_{R'}$ be the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R' and α the element of $\mathcal{R}$ such that
$$\mathfrak{a} = \mathfrak{a}'\oplus\mathfrak{a}^\alpha .$$
For δ in $\mathcal{R}$ denote again by δ the character of $\mathbf{T}$ whose differential at the identity is $\mathrm{ad}\,x \longmapsto \delta(x)$.

Setting:
$$\mathfrak{G}_{d-1,d,d,d+1} := \mathrm{Gr}_{d-1}(\mathfrak{r})\times\mathrm{Gr}_d(\mathfrak{r})\times\mathrm{Gr}_d(\mathfrak{r})\times\mathrm{Gr}_{d+1}(\mathfrak{r}) \quad\text{and}\quad \mathfrak{G}_{d-1,d,d+1} := \mathrm{Gr}_{d-1}(\mathfrak{r})\times\mathrm{Gr}_d(\mathfrak{r})\times\mathrm{Gr}_{d+1}(\mathfrak{r}),$$
denote by θ_α and θ'_α the maps
$$\Bbbk\times A' \xrightarrow{\theta_\alpha} \mathfrak{G}_{d-1,d,d,d+1}\ , \quad (z,g)\longmapsto (g.\mathfrak{t}_\alpha, g.\mathfrak{t}, g\exp(z\mathrm{ad}\,x_\alpha).\mathfrak{t}, g.(\mathfrak{t}+\mathfrak{a}^\alpha)),$$
$$A' \xrightarrow{\theta'_\alpha} \mathfrak{G}_{d-1,d,d+1}\ , \quad g\longmapsto (g.\mathfrak{t}_\alpha, g.\mathfrak{t}, g.(\mathfrak{t}+\mathfrak{a}^\alpha)).$$
Let I_α and S_α be the closures in $\mathrm{Gr}_{d-1}(\mathfrak{r})$ and $\mathrm{Gr}_{d+1}(\mathfrak{r})$ of the orbits of $\mathfrak{t}_\alpha$ and $\mathfrak{t}+\mathfrak{a}^\alpha$ under A' respectively.

Lemma 2.13. *Let Γ and Γ' be the closures in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$ of the images of θ_α and θ'_α.*

(i) *The varieties Γ and Γ' have dimension n and $n-1$ respectively. Moreover, they are invariant under the diagonal actions of R' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$.*

(ii) *The image of Γ by the first, second, third and fourth projections are equal to I_α, $X_{R'}$, X_R, S_α respectively.*

(iii) *The set Γ' is the image of Γ by the projection*
$$\mathfrak{G}_{d-1,d,d,d+1} \xrightarrow{\varpi} \mathfrak{G}_{d-1,d,d+1}\ , \qquad (V_1,V',V,W)\longmapsto (V_1,V',W).$$

(iv) *For all (V_1,V',V,W) in Γ, V_1 is contained in $V'\cap V$ and $V'+V$ is contained in W.*

(v) *Let (V_1,V',V,W) be in Γ such that V' is contained in $\mathfrak{t}_\alpha+\mathfrak{a}'$. Then W is contained in $\mathfrak{t}_\alpha+\mathfrak{a}$.*

(vi) *Let (V_1,V',V,W) be in Γ such that V' is not contained in $\mathfrak{t}_\alpha+\mathfrak{a}$. Then W is not commutative.*

Proof. (i) The maps θ_α and θ'_α are injective since $\mathfrak{t}$ is the normalizer of $\mathfrak{t}$ in $\mathfrak{r}$ by Condition (1), whence $\dim\Gamma = n$ and $\dim\Gamma' = n-1$. For (z,g,k) in $\Bbbk\times A'\times A'$, $\theta_\alpha(z,kg) = k.\theta_\alpha(z,g)$ and $\theta'_\alpha(kg) = k.\theta'_\alpha(g)$. Hence

Γ and Γ' are invariant under the diagonal action of A' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$. Let k be in $\mathbf{T}$. For all (z,g) in $\Bbbk \times A'$,

$$kg.\mathfrak{t}_\alpha = kgk^{-1}(\mathfrak{t}_\alpha), \qquad kg.\mathfrak{t} = kgk^{-1}(\mathfrak{t}),$$

$$kg.(\mathfrak{t} + \mathfrak{a}^\alpha) = kgk^{-1}.(\mathfrak{t} + \mathfrak{a}^\alpha), \qquad kg\exp(z\operatorname{ad} x_\alpha).\mathfrak{t} = kgk^{-1}\exp(z\alpha(k)\operatorname{ad} x_\alpha).\mathfrak{t}$$

so that the images of θ_α and θ'_α are invariant under $\mathbf{T}$, whence the assertion.

(ii) Since $\mathrm{Gr}_d(\mathfrak{r})$, $\mathrm{Gr}_{d-1}(\mathfrak{r})$, $\mathrm{Gr}_{d+1}(\mathfrak{r})$ are projective varieties, the images of Γ by the first, second, third and fourth projections are closed subsets of their target varieties. Since the image of θ_α is contained in the closed subset $I_\alpha \times X_{R'} \times X_R \times S_\alpha$ of $\mathfrak{G}_{d-1,d,d,d+1}$, they are contained in I_α, $X_{R'}$, X_R and S_α respectively. By definition, $R'.\mathfrak{t}_\alpha$, $R'.\mathfrak{t}$ and $R'.(\mathfrak{t} + \mathfrak{a}^\alpha)$ are contained in the images of Γ by the first, second and fourth projections and $R.\mathfrak{t}$ is contained in the image of Γ by the third projection since A is the image of $\Bbbk \times A'$ by the map $(z,g) \mapsto g\exp(z\operatorname{ad} x_\alpha)$, whence the assertion.

(iii) As $\mathrm{Gr}_d(\mathfrak{r})$ is a projective variety, $\varpi(\Gamma)$ is a closed subset of $\mathfrak{G}_{d-1,d,d+1}$ containing the image of θ'_α since $\varpi\circ\theta_\alpha(z,g) = \theta'_\alpha(g)$ for all (z,g) in $\Bbbk \times A'$. Moreover, Γ is contained in $\varpi^{-1}(\Gamma')$, whence $\Gamma' = \varpi(\Gamma)$.

(iv) The subset $\widetilde{\Gamma}$ of elements (V_1, V', V, W) of $\mathfrak{G}_{d-1,d,d,d+1}$ such that V_1 is contained in V' and V and such that V' and V are contained in W, is closed. For all (z,g) in $\Bbbk \times A'$,

$$g\exp(z\operatorname{ad} x_\alpha).(\mathfrak{t} + \mathfrak{a}^\alpha) = g.(\mathfrak{t} + \mathfrak{a}^\alpha).$$

Hence the image of θ_α and Γ are contained in $\widetilde{\Gamma}$ so that V_1 and $V + V'$ are contained in $V' \cap V$ and W respectively for all (V_1, V', V, W) in Γ.

(v) Denote by Γ_* the closure in $\mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ of the image of the map

$$A' \xrightarrow{\theta_{\alpha,*}} \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) , \qquad g \longmapsto (g(\mathfrak{t}), g(\mathfrak{t} + \mathfrak{a}^\alpha)).$$

For all (T_1, T', T, T_2) in the image of θ_α, (T', T_2) is in the image of $\theta_{\alpha,*}$. Then Γ_* is the image of Γ by the projection

$$\mathfrak{G}_{d-1,d,d,d+1} \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) , \qquad (T_1, T', T, T_2) \longmapsto (T', T_2).$$

Denote by τ the quotient morphism

$$\mathfrak{r} \xrightarrow{\tau} \mathfrak{r}/\mathfrak{a}' = \mathfrak{t} + \mathfrak{a}^\alpha .$$

For g in A' and x in $\mathfrak{r}$, $\tau\circ g(x) = \tau(x)$. Set:

$$X := \{(g, t, z, z', v, w) \in A' \times \mathfrak{t}_\alpha \times \Bbbk^2 \times \mathfrak{r}' \times \mathfrak{r}; \mid v = g(zs + t),\ w = g(zs + t + z'x_\alpha)\}$$

and denote by Y the closure in $\mathfrak{r}' \times \mathfrak{r}$ of the image of X by the canonical projection

$$A' \times \mathfrak{t}_\alpha \times \Bbbk^2 \times \mathfrak{r}' \times \mathfrak{r} \longrightarrow \mathfrak{r}' \times \mathfrak{r} .$$

As for all (g, t, z, z', v, w) in X,

$$\tau(v) = zs + t \quad \text{and} \quad \tau(w) = zs + t + z'x_\alpha,$$

$$\alpha\circ\pi\circ\tau(v) = \alpha\circ\pi\circ\tau(w)$$

for all (v,w) in Y. By definition, for all (T, T') in Γ_*, $T \times T'$ is contained in Y. By hypothesis, V' is contained in the kernel of $\alpha\circ\pi$ and (V', W) is in Γ_*. Hence W is contained in the kernel of $\alpha\circ\pi$.

(vi) Denote by Γ'_* the closure in $\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r})$ of the image of the map

$$\Bbbk \times A' \xrightarrow{\theta'_{\alpha,*}} \mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) , \qquad (z,g) \longmapsto (\theta_\alpha(z,g), g(\mathfrak{a}^\alpha))$$

and Γ'_{**} the closure in $\mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r})$ of the image of the map

$$A' \longrightarrow \mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r}) , \qquad g \longmapsto (g(\mathfrak{t}), g(\mathfrak{a}^{\alpha}).$$

For all (T_1, T', T, T_2, T'_2) in the image of $\theta'_{\alpha,*}$, $T' + T'_2$ is contained in T_2. Then so is it for all (T_1, T', T, T_2, T'_2) in Γ'_*. As $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathrm{Gr}_1(\mathfrak{r})$ are projective varieties, Γ and Γ'_{**} are the images of Γ'_* by the projections

$$\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) \longrightarrow \mathfrak{G}_{d-1,d,d,d+1} , \qquad (T_1, T', T, T_2, T'_2) \longmapsto (T_1, T', T, T_2),$$

$$\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) \longrightarrow \mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r}) , \qquad (T_1, T', T, T_2, T'_2) \longmapsto (T', T'_2)$$

respectively.

Set:

$$X' := \{(g, t, z, v, w) \in A' \times \mathfrak{t} \times \Bbbk \times \mathfrak{r}' \times \mathfrak{r} \mid v = g(t), \ w = g(zx_\alpha)\}$$

and denote by Y' the closure in $\mathfrak{r}' \times \mathfrak{r}$ of the image of X' by the canonical projection

$$A' \times \mathfrak{t} \times \Bbbk \times \mathfrak{r}' \times \mathfrak{r} \longrightarrow \mathfrak{r}' \times \mathfrak{r} .$$

As for all (g, t, z, v, w) in X',

$$[v, w] = g([t, zx_\alpha]) = \alpha(t)g(zx_\alpha) = \alpha{\circ}\pi(v)w,$$

$[v, w] = \alpha{\circ}\pi(v)w$ for all (v, w) in Y'. By definition, for all (T, T') in Γ'_{**}, $T \times T'$ is contained in Y'. For some W' in $\mathrm{Gr}_1(\mathfrak{r})$, (V_1, V', V, W, W') is in Γ'_*. By hypothesis, V' is not contained in the kernel of $\alpha{\circ}\pi$. Hence, for some v in V' and w in $W \setminus \{0\}$, $\alpha{\circ}\pi(v) \neq 0$ and $[v, w] = \alpha{\circ}\pi(v)w$. □

Corollary 2.14. *Suppose that $\mathfrak{a}'$ has Property* (**P**)*. Let s be in $\mathfrak{t}$ such that $\mathfrak{r}^s$ is contained in $\mathfrak{a}'$ and (V_1, V', V, W) be in Γ such that V is contained in $\mathfrak{r}^s$ and $[s, V']$ is contained in V'.*

(i) *If W is not commutative then $V' = V$ and V is in $\overline{R^s.\mathfrak{t}}$.*

(ii) *Suppose that for some v in $\mathfrak{a}$, $s + v$ is in V. Then $V' = V$ and V is in $\overline{R^s.\mathfrak{t}}$.*

Proof. By Lemma 2.13(ii), V and V' are in X_R and $X_{R'}$ respectively.

(i) If $V' = V$, V is in $\overline{R^s.\mathfrak{t}}$ by Property (**P**) for $\mathfrak{a}'$. Suppose $V' \neq V$. A contradiction is expected. Then, by Lemma 2.13(iv), for some x and y in W,

$$V = V_1 \oplus \Bbbk x, \quad V' = V_1 \oplus \Bbbk y, \quad W = V_1 \oplus \Bbbk x \oplus \Bbbk y.$$

Moreover, as V is contained in $\mathfrak{r}^s$ and $[s, V'] \subset V'$, W is contained in $\mathfrak{r}'$ and we can choose y so that $[s, y] \in \Bbbk y$. Since V and V' are commutative subalgebras of $\mathfrak{r}$, $[x, y] \neq 0$. We have two cases to consider:

(a,1) V' is contained in $\mathfrak{r}^s$,

(a,2) V' is not contained in $\mathfrak{r}^s$.

(a,1) By Property (**P**) for $\mathfrak{a}'$, s is in V'. Hence $s = ty + v$ for some in (t, v) in $\Bbbk \times V_1$. As V is a commutative subalgebra of $\mathfrak{r}^s$, containing V_1 and x,

$$0 = [x, s] = t[x, y].$$

Then $s = v$ is in V_1, whence a contradiction since $\alpha(s) \neq 0$ and V_1 is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ by Lemma 2.13(ii).

(a,2) For some a in $\Bbbk^*$, $[s, y] = ay$. Then y is in $\mathfrak{a}'$ and V' is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since so is V_1. As a result, by Lemma 2.13(v), V and W are contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since V is contained in $\mathfrak{a}'$. As $[s, [x, y]] = a[x, y]$, $[x, y] = by$ for some b in $\Bbbk^*$ since the eigenspace of eigenvalue a of the restriction of $\mathrm{ad}\, s$ to V' is generated by y. Then $\mathrm{ad}\, x$ is not nilpotent. Let x_{s} be in $\mathfrak{r}'$ such that $\mathrm{ad}\, x_{\mathrm{s}}$ is the semisimple component of $\mathrm{ad}\, x$. Then x_{s} is in $\mathfrak{t}_\alpha + \mathfrak{a}'$, $[s, x_s] = 0$ and $[x_{\mathrm{s}}, V_1] = \{0\}$ since $[s, x] = 0$ and $[x, V_1] = \{0\}$. Moreover, $[ax_{\mathrm{s}} - bs, y] = 0$. Then $ax_{\mathrm{s}} - bs$ is a semisimple element of $\mathfrak{r}'$ such that $[ax_{\mathrm{s}} - bs, V'] = \{0\}$. As it is conjugate under R' to an element of $\mathfrak{t}$ by Lemma 2.1(ii), by Property (**P**) for $\mathfrak{a}'$, $ax_{\mathrm{s}} - bs$ is in V', whence a contradiction since V' is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ and $ax_{\mathrm{s}} - bs$ is not in $\mathfrak{t}_\alpha + \mathfrak{a}'$.

(ii) If $V = V'$, V is in $\overline{R^s.\mathfrak{t}}$ by Property (**P**) for $\mathfrak{a}'$. Suppose $V \neq V'$. A contradiction is expected. As V is contained in $\mathfrak{r}^s$, $[s, v] = 0$. Let x and y be as in (i). As V_1 is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$, $s + v$ is not in V_1 since $\alpha(s) \neq 0$. So we can choose $s+v = x$. By (i), W is commutative. Then $[s+v, y] = 0$ and $[\mathrm{ad}\, s, \mathrm{ad} y] = 0$ since $\mathrm{ad}\, s$ is the semisimple component of $\mathrm{ad}(s + v)$. Hence, by Lemma 2.1(i), $[s, y] = 0$ since $[s, y]$ is in $\mathfrak{a}$. As a result, V' is contained in $\mathfrak{r}^s$ since so is V_1. So, by Property (**P**) for $\mathfrak{a}'$, s is in V' and W is not commutative by Lemma 2.13(vi), whence a contradiction. □

For (T_1, T', T_2) in Γ', denote by Γ_{T_1,T',T_2} the subset of elements (T_1, T', T, T_2) of $\mathfrak{G}_{d-1,d,d,d+1}$ such that T is contained in T_2 and contains T_1. Then Γ_{T_1,T',T_2} is a closed subvariety of $\mathfrak{G}_{d-1,d,d,d+1}$, isomorphic to $\mathbb{P}^1(\Bbbk)$. Let (V_1, V', V, W) be a fixed point under $\mathbf{T}$ of Γ.

Lemma 2.15. (i) *For some affine open neighborhood Ω of (V_1, V', W) in Γ', Ω is invariant under* $\mathbf{T}$.

(ii) *For $i = 0, \ldots, n-2$, there exist Y_i and O_i such that*

- (a) *Y_i is an irreducible closed subset of dimension $n-1-i$ of Ω, containing (V_1, V', W) and invariant under* $\mathbf{T}$*,*
- (b) *O_i is a locally closed subvariety of dimension $n-1-i$ of A', invariant under $\mathbf{T}$ by conjugation,*
- (c) *$\theta'_\alpha(O_i)$ is contained in Y_i and (V_1, V', V, W) is in the closure of $\theta_\alpha(\Bbbk \times O_i)$ in Γ.*

(iii) *There exist a smooth projective curve C, an action of $\mathbf{T}$ on C, $x_1, \ldots, x_m$ in C and two morphisms*

$$C \setminus \{x_1, \ldots, x_m\} \xrightarrow{\ \mu\ } A' \ , \qquad C \xrightarrow{\ \nu\ } \Gamma'$$

satisfying the following conditions:

- (a) *$x_1, \ldots, x_m$ are the fixed points under $\mathbf{T}$ in C,*
- (b) *for g in $\mathbf{T}$ and x in $C \setminus \{x_1, \ldots, x_m\}$, $\mu(g.x) = g\mu(x)g^{-1}$ and $\nu(g.x) = g.\nu(x)$,*
- (c) *$\nu(x_1) = (V_1, V', W)$,*
- (d) *(V_1, V', V, W) is in the closure of the image of $\Bbbk \times (C \setminus \{x_1, \ldots, x_m\})$ by the map $(z, x) \longmapsto \theta_\alpha(z, \mu(x))$.*

Proof. (i) As Γ' is a projective variety with a $\mathbf{T}$ action and (V_1, V', W) is a fixed point under $\mathbf{T}$, there exists an affine open neighborhood Ω of (V_1, V', W) in Γ', invariant under $\mathbf{T}$.

(ii) Prove the assertion by induction on i. For $i = 0$, $Y_i = \Omega$ and O_i is the inverse image of Ω by θ'_α. Suppose that Y_i and O_i are known. Let Y'_i be the closure in Γ of $\theta_\alpha(\Bbbk \times O_i)$. By Condition (c), Y'_i is invariant under $\mathbf{T}$ and $\theta_\alpha(\Bbbk \times O_i)$ is a $\mathbf{T}$-invariant dense subset of Y'_i. So, it contains an $\mathbf{T}$-invariant dense open subset O'_i of Y'_i. As θ'_α is an orbital injective morphism, $\theta'_\alpha(O_i)$ is a dense open subset of Y_i. Set:

$$Z' := Y'_i \setminus O'_i, \quad Z := Y_i \setminus \theta_\alpha(O'_i), \quad Z_* := \Omega \cap (\varpi(Z) \cup Z').$$

Then Z_* is a $\mathbf{T}$-invariant closed subset of Y_i, containing (V_1, V', W).

Denote by Z_{**} the union of irreducible components of dimension $\dim Y_i - 1$ of Z_* and I the union of the ideals of definition in $\Bbbk[Y_i]$ of the irreducible components of Z_{**}. Let p be in $\Bbbk[Y_i] \setminus I$, semiinvariant under $\mathbf{T}$ and such that $p((V_1, V', W)) = 0$. Denote by Y'_{i+1} an irreducible component of the nullvariety of p in $Y'_i \cap \varpi^{-1}(\Omega)$, containing (V_1, V', V, W) and Y_{i+1} the closure in Ω of $\varpi(Y'_{i+1})$. Then Y_{i+1} has dimension $n-i-1$ and its intersection with $\theta'_\alpha(O_i)$ is not empty so that $O_{i+1} := \theta'^{-1}_\alpha(O_i \cap \theta'_\alpha(O_i))$ is a nonempty locally closed subset of dimension $n-i-1$ of A'. Moreover, Y_{i+1} and O_{i+1} are invariant under $\mathbf{T}$ since p is semiinvariant under $\mathbf{T}$. As $\theta_\alpha(\Bbbk \times O_{i+1})$ is the intersection of Y'_{i+1} and $\theta_\alpha(\Bbbk \times O_i)$, it is dense in Y'_{i+1} so that (V_1, V', V, W) is in the closure of $\theta_\alpha(\Bbbk \times O_{i+1})$ and (V_1, V', W) is in Y_{i+1}.

(iii) Let $\overline{Y_{n-2}}$ be the closure of Y_{n-2}, C its normalization and ν the normalization morphism. Then C is a smooth projective curve. As Y_{n-2} is invariant under $\mathbf{T}$, so is $\overline{Y_{n-2}}$ and there is an action of $\mathbf{T}$ on C such that ν is an equivariant morphism. As the restriction of θ'_{α} to O_{n-2} is an isomorphism onto a dense open subset of Y_{n-2}, the actions of $\mathbf{T}$ on $\overline{Y_{n-2}}$ and C are not trivial since θ'_{α} is equivariant under the actions of $\mathbf{T}$. As a result, $\mathbf{T}$ has an open orbit O_* in $\overline{Y_{n-2}}$ and $\overline{Y_{n-2}} \setminus O_*$ is the set of fixed points under $\mathbf{T}$ of $\overline{Y_{n-2}}$ since $\overline{Y_{n-2}}$ has dimension 1. Hence the restriction of ν to $\nu^{-1}(O_*)$ is an isomorphism, $C \setminus \nu^{-1}(O_*)$ is finite, its elements are fixed under $\mathbf{T}$ and there exists a $\mathbf{T}$-equivariant morphism μ from $\nu^{-1}(O_*)$ to A' such that $\theta'_{\alpha}\circ\mu = \nu$. As (V_1, V', W) is a fixed point under $\mathbf{T}$, for some x_1 in $C \setminus \nu^{-1}(O_*)$, $\nu(x_1) = (V_1, V', W)$ since (V_1, V', W) is in $\nu(C)$. Moreover, (V_1, V', V, W) is in the closure of the map

$$\Bbbk \times (C \setminus \nu^{-1}(O_*)) \longrightarrow \Gamma \ , \qquad (z, x) \longmapsto \theta_{\alpha}(z, \mu(x))$$

since it is in $\overline{\theta_{\alpha}(\Bbbk \times O_{n-2})}$. □

Denote by η the morphism

$$\Bbbk \times (C \setminus \{x_1, \dots, x_m\}) \xrightarrow{\ \eta\ } \Gamma \ , \qquad (z, x) \longmapsto \theta_{\alpha}(z, \mu(x))$$

and Δ the closure of the graph of η in $\mathbb{P}^1(\Bbbk) \times C \times \Gamma$. Let υ be the restriction to Δ of the canonical projection

$$\mathbb{P}^1(\Bbbk) \times C \times \Gamma \longrightarrow \mathbb{P}^1(\Bbbk) \times C$$

and for (z, x) in $\mathbb{P}^1(\Bbbk) \times C$, let $F_{z,x}$ be the subset of Γ such that $\{(z, x)\} \times F_{z,x}$ is the fiber of υ at (z, x). We have an action of $\mathbf{T}$ in $\mathbb{P}^1(\Bbbk)$ given by

$$t.z := \begin{cases} \alpha(t)z & \text{if} \quad z \in \Bbbk^* \\ z & \text{if} \quad z \in \{0, \infty\} \end{cases} .$$

Lemma 2.16. *Let Δ_{ν} be the graph of ν.*

(i) *The set Δ_{ν} is the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$.*

(ii) *For t in $\mathbf{T}$ and (z, x, y) in Δ, $t.(z, x, y) := (t.z, t.x, t.y)$ is in Δ.*

(iii) *For (z, x) in $\mathbb{P}^1(\Bbbk) \times C$, η is regular at (z, x) if and only if $F_{z,x}$ has dimension 0. In this case, $|F_{z,x}| = 1$.*

(iv) *For (z, x) in $\mathbb{P}^1(\Bbbk) \times C \setminus \{0, \infty\} \times \{x_1, \dots, x_m\}$, η is regular at (z, x).*

(v) *For $i = 1, \dots, m$, there exists a regular map η_i from $\mathbb{P}^1(\Bbbk)$ to Γ such that $\eta_i(z) = \eta(z, x_i)$ for all z in $\Bbbk^*$. Moreover, its image is contained in $\varpi^{-1}(\{\nu(x_i)\}) \cap \Gamma$.*

Proof. (i) As $\mathbb{P}^1(\Bbbk)$ and $\mathrm{Gr}_d(\mathfrak{r})$ are projective varieties, the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$ is a closed subset of $C \times \Gamma'$ contained in Δ_{ν} since $\varpi\circ\eta(z, x) = \nu(x)$ for all (z, x) in $\Bbbk \times (C \setminus \{x_1, \dots, x_m\})$, whence the assertion since the inverse image of Δ_{ν} by this map is a closed subset of $\mathbb{P}^1(\Bbbk) \times C \times \Gamma$ containing the graph of η.

(ii) From the equality

$$t \exp(z \operatorname{ad} x_{\alpha}) t^{-1} = \exp(\alpha(t) z \operatorname{ad} x_{\alpha})$$

for all (t, z) in $\mathbf{T} \times \Bbbk$, we deduce the equality

$$t.\eta(z, x) = t.\theta_{\alpha}(z, \mu(x)) = \theta_{\alpha}(\alpha(t)z, \mu(t.x) = \eta(t.z, t.x)$$

for all (t, z, x) in $\mathbf{T} \times \Bbbk \times (C \setminus \{x_1, \dots, x_m\})$ since θ_{α} and μ are $\mathbf{T}$-equivariant, whence the assertion.

(iii) As Γ is a projective variety, υ is a projective morphism. Moreover, it is birational since Δ is the closure of the graph of η. So, by Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of υ are connected of dimension 0 or 1 since $\mathbb{P}^1(\Bbbk) \times C$ is normal of dimension 2. Let (z, x) be in $\mathbb{P}^1(\Bbbk) \times C$ such that $F_{z,x}$ dimension 0. There exists a neighborhood $O_{z,x}$ of (z, x) in $\mathbb{P}^1(\Bbbk) \times C$ such that F_y has dimension 0 for y

in $O_{z,x}$. In other words, the restriction of υ to $\upsilon^{-1}(O_{z,x})$ is a quasi finite morphism. Moreover, it is birational and surjective. So, again by Zariski's Main Theorem [Mu88, §9], it is an isomorphism. Hence η is regular at (z,x). Conversely, if η is regular at (z,x), $\eta(z,x)$ is an isolated point in $F_{z,x}$, whence $F_{z,x} = \{\eta(z,x)\}$ since $F_{z,x}$ is connected.

(iv) The variety $\Bbbk\times(C\setminus\{x_1,\ldots,x_m\})$ is an open subset of the smooth variety $\mathbb{P}^1(\Bbbk)\times C$ and Γ is a projective variety. Hence η has a regular extension to a big open subset of $\mathbb{P}^1(\Bbbk)\times C$ by [Sh94, Ch. 6, Theorem 6.1]. By Condition (a) of Lemma 2.15(iii), $\{0,\infty\}\times\{x_1,\ldots,x_m\}$ is the set of fixed points under $\mathbf{T}$ in $\mathbb{P}^1(\Bbbk)\times C$ and by (ii), $t.\eta(z,x) = \eta(t.z,t.x)$ for all (t,z,x) in $\mathbf{T}\times\mathbb{P}^1(\Bbbk)\times(C\setminus\{x_1,\ldots,x_m\})$. Hence η is regular on $P^1(\Bbbk)\times C\setminus\{0,\infty\}\times\{x_1,\ldots,x_m\}$.

(v) The restriction of η to $\Bbbk^*\times\{x_i\}$ is a regular map from a dense open subset of the smooth variety $\mathbb{P}^1(\Bbbk)\times\{x_i\}$ to the projective variety Γ. So, again by [Sh94, Ch. 6, Theorem 6.1], this map has regular extension to $\mathbb{P}^1(\Bbbk)\times\{x_i\}$, whence the assertion by (i). □

Let I be the set of indices such that $\nu(x_i) = (V_1, V', W)$. Denote by S the image of Δ by the canonical projection $\mathbb{P}^1(\Bbbk)\times C\times\Gamma \longrightarrow \Gamma$, $S_{\rm n}$ its normalization, σ the normalization morphism, $S^{\mathbf{T}}$ and $S_{\rm n}^{\mathbf{T}}$ the sets of fixed points under $\mathbf{T}$ in S and $S_{\rm n}$ respectively. Set

$$C_* := \mathbb{P}^1(\Bbbk)\times C\setminus\{(0,\infty\}\times\{x_1,\ldots,x_m\}.$$

By Lemma 2.15(iv), η is a dominant morphism from C_* to S, whence a commutative diagram

$$\begin{tikzcd} C_* \arrow[r, "\eta_{\rm n}"] \arrow[dr, "\eta"'] & S_{\rm n} \arrow[d, "\sigma"] \\ & S \end{tikzcd}$$

since C_* is smooth. Let $\Delta_{\rm n}$ be the closure in $\mathbb{P}^1(\Bbbk)\times C\times S_{\rm n}$ of the graph of $\eta_{\rm n}$ and υ_2 the restriction to $\Delta_{\rm n}$ of the canonical projection

$$\mathbb{P}^1(\Bbbk)\times C\times S_{\rm n} \longrightarrow S_{\rm n}\ .$$

Lemma 2.17. *Suppose $V'\neq V$ and V and V' contained in $\mathfrak{z}+\mathfrak{a}$.*

(i) *The variety Δ is the image of $\Delta_{\rm n}$ by the map $(z,x,y)\mapsto(z,x,\sigma(y))$.*

(ii) *The morphism υ_2 is projective and birational.*

(iii) *There exists a $\mathbf{T}$-equivariant morphism*

$$(S_{\rm n}\setminus S_{\rm n}^{\mathbf{T}}) \xrightarrow{\ \varphi\ } C_*$$

such that $\eta\circ\varphi$ is the restriction of σ to $S_{\rm n}\setminus S_{\rm n}^{\mathbf{T}}$.

(iv) *For some i in I, $\eta_i(1)$ is not invariant under $\mathbf{T}$.*

Proof. (i) As S is a projective variety, so are $S_{\rm n}$, $\mathbb{P}^1(\Bbbk)\times C\times S_{\rm n}$, $\Delta_{\rm n}$ and the image of $\Delta_{\rm n}$ by the map $(z,x,y)\mapsto(z,x,\sigma(y))$, whence the assertion since the image of the graph of $\eta_{\rm n}$ by this map is the graph of η.

(ii) As $\Delta_{\rm n}$ is projective so is υ_2. Since θ_α is injective, so is the restriction of η to $\Bbbk\times(C\setminus\{x_1,\ldots,x_m\})$. Hence υ_2 is birational.

(iii) By (ii) and Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of υ_2 are connected. For y in $S_{\rm n}\setminus S_{\rm n}^{\mathbf{T}}$ and (z,x) in $\mathbb{P}^1(\Bbbk)\times C$ such that (z,x,y) is in $\Delta_{\rm n}$, $\varpi\circ\sigma(y) = \nu(x)$ by (i). If x is not in $\{x_1,\ldots,x_m\}$, $\nu^{-1}(\varpi\circ\sigma(y)) = \{x\}$ by Condition (b) of Lemma 2.15(iii) and z is the element of $\Bbbk$ such that $\theta_\alpha(z,\mu(x)) = \sigma(y)$. Suppose $x = x_i$ for some $i = 1,\ldots,m$. Let z and z' be in $\Bbbk^*$ such that (z,x_i,y) and (z',x_i,y) are in $\Delta_{\rm n}$. Then $(z,x_i,\sigma(y))$ and $(z',x_i,\sigma(y))$ are in Δ. By Lemma 2.16,(iii) and (iv), $\sigma(y) = \eta(z,x_i) = \eta(z',x_i)$. For some t

in $\mathbf{T}$, $z' = t.z$ so that $t.\sigma(y) = \sigma(y)$. As y is not invariant under $\mathbf{T}$ so is $\sigma(y)$ since the fibers of σ are finite. Hence the stabilizer of $\sigma(y)$ in $\mathbf{T}$ is finite and so is the fiber of υ_2 at y. As a result, the restriction of υ_2 to $\Delta_{\rm n} \setminus \mathbb{P}^1(\Bbbk) \times C \times S_{\rm n}^{\mathbf{T}}$ is an injective morphism. So, again by Zariski's Main Theorem [Mu88, §9], this map is an isomorphism, whence a morphism

$$(S_{\rm n} \setminus S_{\rm n}^{\mathbf{T}}) \xrightarrow{\varphi} C_* \ .$$

Moreover, φ is $\mathbf{T}$-equivariant since so is υ_2. For y in $S_{\rm n}$ such that $\sigma(y) = \eta(z,x)$ for some (z,x) in $\Bbbk^* \times (C \setminus \{x_1,\dots,x_m\})$, (z,x,y) is the unique element of $\Delta_{\rm n}$ above y. Hence $\eta\circ\varphi = \sigma$.

(iv) Suppose that for all i in I, $\eta_i(1)$ is invariant under $\mathbf{T}$. A contradiction is expected. As $V \neq V'$, $V_1 = V \cap V'$ and $V + V' = W$ by Lemma 2.13(iv). Moreover, since V and V' are contained in $\mathfrak{z} + \mathfrak{a}$, for some β and γ in $\mathcal{R}$,

$$V = V_1 + \mathfrak{a}^{\beta} \quad \text{and} \quad V' = V_1 + \mathfrak{a}^{\gamma}.$$

Then $\Gamma_{V_1,V',W}$ is invariant under $\mathbf{T}$. More precisely, $\Gamma_{V_1,V',W}$ is a union of one orbit of dimension 1 and the set $\{(V_1,V',V',W),(V_1,V',V,W)\}$ of fixed points. As a result, $\Gamma_{V_1,V',W}\cap S$ is equal to $\{(V_1,V',V',W),(V_1,V',V,W)\}$ or $\Gamma_{V_1,V',W}$ since S is invariant under $\mathbf{T}$. By Lemma 2.16,(ii) and (v), for i in I, $\eta_i(\mathbb{P}^1(\Bbbk))$ is equal to (V_1,V',V',W) or (V_1,V',V,W) since $\nu(x_i) = (V_1,V',W)$.

Suppose $\Gamma_{V_1,V',W} \cap S = \{(V_1,V',V',W),(V_1,V',V,W)\}$. By Lemma 2.16,(v) and (iii), for all i in I, η is regular at $(0,x_i)$ and (∞,x_i) since $\nu(x_i) = (V_1,V',W)$, whence

$$\lim_{z\to 0} \eta_i(0) = (V_1,V',V',W) \quad \text{and} \quad \lim_{z\to\infty} \eta_i(\infty) = (V_1,V',V,W),$$

whence a contradiction.

Suppose $\Gamma_{V_1,V',W} \cap S = \Gamma_{V_1,V',W}$. Let y be in $S_{\rm n}$ such that

$$\sigma(y) \in \Gamma_{V_1,V',W} \setminus \{(V_1,V',V',W),(V_1,V',V,W)\}.$$

By (iii), for some i in I and some z in $\Bbbk^*$, $\varphi(t.y) = (t.z,x_i)$ and $t.\sigma(y) = t.\eta(z,x_i) = t.\eta_i(z)$ for all t in $\mathbf{T}$, whence a contradiction since (V_1,V',V',W) and (V_1,V',V,W) are in $\overline{\mathbf{T}.\sigma(y)}$. □

Corollary 2.18. *Let (V_1,V',V,W) be a fixed point under $\mathbf{T}$ of Γ such that $V\cap\mathfrak{t} = V'\cap\mathfrak{t} = \mathfrak{z}$. Then $V' = V$.*

Proof. Suppose $V' \neq V$. A contradiction is expected. By Lemma 2.13(iv), $V_1 = V\cap V'$ and $W = V + V'$. As $V\cap\mathfrak{t} = V'\cap\mathfrak{t} = \mathfrak{z}$, V and V' are contained in $\mathfrak{z}+\mathfrak{a}$. So, for some β in $\mathcal{R}$ and γ in $\mathcal{R}\setminus\{\alpha\}$,

$$V = V_1 \oplus \mathfrak{a}^{\beta} \quad \text{and} \quad V' = V_1 \oplus \mathfrak{a}^{\gamma}.$$

since (V_1,V',V,W) is invariant under $\mathbf{T}$. By Lemma 2.17(iv), for some i in I, $\eta_i(1)$ is not fixed under $\mathbf{T}$. Then, by Lemma 2.13(ii), $\eta_i(\mathbb{P}^1(\Bbbk)) = \Gamma_{V_1,V',W}$. Denoting by $\eta_i(z)_3$ the third component of $\eta_i(z)$, for all z in $\mathbb{P}^1(\Bbbk)$, V_1 is contained in $\eta_i(z)_3$ and $\eta_i(z)_3$ is contained in W. Hence for some a in $\Bbbk^*$,

$$\eta_i(1)_3 = V_1 \oplus \Bbbk(x_\beta + a x_\gamma) \quad \text{and} \quad \eta_i(\alpha(t))_3 = V_1 \oplus \Bbbk(\beta(t)x_\beta + \gamma(t)a x_\gamma)$$

for all t in $\mathbf{T}$. For some t_1 and t_2 in $\mathbf{T}$, for all δ in $\mathcal{R}$, $\delta(t_1)$ and $\delta(t_2)$ are positive rational numbers and

$$\alpha(t_1) > 1, \qquad \alpha(t_2) > 1, \qquad \beta(t_1) < \gamma(t_1), \qquad \beta(t_2) > \gamma(t_2).$$

Then

$$\lim_{k\to\infty} V_1 \oplus \Bbbk(\beta(t_1^k)x_\beta + \gamma(t_1^k)a x_\gamma) = V_1 \oplus \mathfrak{a}^{\gamma}, \quad \lim_{k\to\infty} V_1 \oplus \Bbbk(\beta(t_2^k)x_\beta + \gamma(t_2^k)a x_\gamma) = V_1 \oplus \mathfrak{a}^{\beta},$$

$$\lim_{k\to\infty} \eta_i(\alpha(t_1^k) = \lim_{k\to\infty} \eta_i(\alpha(t_2^k) = \eta_i(\infty),$$

whence $V = V'$ and the contradiction. □

2.6. **Property (P) and Property (P$_1$).** In this subsection we suppose that all objects of $\mathcal{C}'_{\mathfrak{t}}$ of dimension smaller than n has Property (**P**). For V a fixed point of X_R under $\mathbf{T}$, denote by Λ_V the orthogonal complement to $\mathfrak{z}_V$ in $\mathcal{R}$ and set:

$$\mathfrak{r}_V := \mathfrak{r}_{\Lambda_V}, \qquad R_V := R_{\Lambda_V}.$$

Lemma 2.19. *Let V be a fixed point under* $\mathbf{T}$ *in X_R.*

(i) *The action of R_V in $\overline{R_V.V}$ has fixed points. For V_∞ such a point,*

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta, \quad |\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|, \quad \mathrm{r}_V \geqslant \mathrm{r}_{V_\infty}.$$

(ii) *The set $\mathcal{R}_V$ has rank at least $|\mathcal{R}_V| - 1$.*

(iii) *Suppose that $\mathfrak{a}$ has Property* (**P**$_1$). *Then $\mathcal{R}_V$ has rank $|\mathcal{R}_V|$.*

(iv) *If $\mathfrak{a}$ has Property* (**P**$_1$)*, for s in $\mathfrak{t}$ such that V is contained in $\mathfrak{r}^s$, V is in $\overline{R^s.\mathfrak{t}}$.*

Proof. (i) As $\overline{R_V.V}$ is a projective variety and R_V is connected and solvable, R_V has fixed points in $\overline{R_V.V}$. Denote by V_∞ such a point. Since V is fixed under $\mathbf{T}$,

$$V = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta.$$

Moreover, $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$ since V is commutative. By Lemma 2.6(ii), $\mathfrak{z}_V$ is the center of $\mathfrak{r}_V$. Hence $V \cap \mathfrak{t}$ is contained in all element of $R_V.V$. Moreover, all element of $R_V.V$ is contained in $V \cap \mathfrak{t} + \mathfrak{a}_{\Lambda_V}$. Then

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta,$$

whence $|\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|$. Since $\mathcal{R}_{V_\infty}$ is contained in Λ_V and $\mathrm{r}_V = d - \dim \mathfrak{z}_V$, $\mathrm{r}_V \geqslant \mathrm{r}_{V_\infty}$.

(ii) By (i), we can suppose that V is invariant under R_V. By Lemma 2.5, $\mathfrak{a}_{\Lambda_V}$ is contained in an ideal $\mathfrak{a}'$ of $\mathfrak{r}$ of dimension $\dim \mathfrak{a} - 1$ and contained in $\mathfrak{a}$. We then use the notations of Lemma 2.13. Set $\Gamma_V := \varpi_3^{-1}(V)$. By Lemma 2.13(i), Γ_V is a projective variety invariant under R_V since so is V. Then R_V has a fixed point in Γ_V. Let (V_1, V', V, W) be such a point. As $\mathfrak{a}'$ has Property (**P**), by Lemma 2.12(i),

$$V' = \mathfrak{z}_{V'} \oplus \bigoplus_{\beta \in \mathcal{R}_{V'}} \mathfrak{a}^\beta.$$

and the elements of $\mathcal{R}_{V'}$ are linearly independent.

If $V' = V$ then $\mathcal{R}_{V'} = \mathcal{R}_V$ so that $\mathrm{r}_V = \mathrm{r}_{V'} = |\mathcal{R}_V|$. Suppose $V' \neq V$. Then, by Lemma 2.13(iv),

$$V_1 = \mathfrak{z}_{V'} \cap V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V \cap \mathcal{R}_{V'}} \mathfrak{a}^\beta.$$

As V_1 has codimension 1 in V and V', $\mathcal{R}_{V'} = \mathcal{R}_V$ or $\mathfrak{z}_{V'} = V \cap \mathfrak{t}$. In the first case, $\mathrm{r}_V = |\mathcal{R}_V|$ and in the second case,

$$|\mathcal{R}_V \cap \mathcal{R}_{V'}| = |\mathcal{R}_V| - 1 = |\mathcal{R}_{V'}| - 1,$$

whence $\mathrm{r}_V \geqslant |\mathcal{R}_V| - 1$ since the elements of $\mathcal{R}_{V'}$ are linearly independent.

(iii) Prove the assertion by induction on $\dim \mathfrak{z}_V$. If $\mathfrak{z}_V = \mathfrak{z}$, then $r_V = |\mathcal{R}_V|$ by Property (**P**$_1$). Suppose $\dim \mathfrak{z}_V = \dim \mathfrak{z} + 1$ and $V \cap \mathfrak{t} = \mathfrak{z}$. Then $|\mathcal{R}_V| = d^\#$ and $r_V = d^\# - 1$. By Property (**P**$_1$), it is impossible. Hence $V \cap \mathfrak{t} = \mathfrak{z}_V$ since $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$. As a result $r_V = |\mathcal{R}_V|$.

Suppose $\dim \mathfrak{z}_V \geqslant 2 + \dim \mathfrak{z}$, the assertion true for the integers smaller than $\dim \mathfrak{z}_V$ and $r_V < |\mathcal{R}_V|$. A contradiction is expected. By (ii), $V \cap \mathfrak{t}$ has dimension at least $\dim \mathfrak{z}_V - 1$. Then, for some α in $\mathcal{R}$, $V \cap \mathfrak{t}_\alpha$ is strictly contained in $V \cap \mathfrak{t}$. Let Λ be the orthogonal complement to $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ in $\mathcal{R}$. As $\overline{R_\Lambda.V}$ is a projective variety and R_Λ is connected, R_Λ has a fixed point in $\overline{R_\Lambda.V}$. Let V_∞ be such a point. By Lemma 2.6(ii),

$\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of $\mathfrak{r}_\Lambda$. Hence $V \cap \mathfrak{t}_\alpha$ is contained in all element of $R_\Lambda.V$. Moreover, all element of $R_\Lambda.V$ is contained in $V \cap \mathfrak{t} + \mathfrak{a}_\Lambda$. As V_∞ is an ideal of $\mathfrak{r}_\Lambda$, $V \cap \mathfrak{t}$ is not contained in V_∞ since it is not contained in the kernel of α. Then

$$V_\infty = V \cap \mathfrak{t}_\alpha \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta .$$

By (ii), $r_{V_\infty} \geqslant |\mathcal{R}_{V_\infty}| - 1$, whence

$$\dim \mathfrak{z}_{V_\infty} \leqslant \dim V \cap \mathfrak{t}_\alpha + 1 = \dim V \cap \mathfrak{t} < \dim \mathfrak{z}_V .$$

So, by induction hypothesis, $|\mathcal{R}_{V_\infty}| = \mathrm{r}_{V_\infty}$ and $\mathfrak{z}_{V_\infty} = V \cap \mathfrak{t}_\alpha$. Since $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of $\mathfrak{r}_\Lambda$, $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is contained in $\mathfrak{z}_{V_\infty}$, whence

$$\dim \mathfrak{z}_V - 1 \leqslant \dim V \cap \mathfrak{t}_\alpha = \dim V \cap \mathfrak{t} - 1 .$$

As a result, $\mathfrak{z}_V = V \cap \mathfrak{t}$ since $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$, whence a contradiction.

(iv) Suppose that $\mathfrak{a}$ has Property ($\mathbf{P}_1$). By (iii),

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta$$

and $\mathrm{r}_V = |\mathcal{R}_V|$. As a result, the centralizer of V in $\mathfrak{t}$ is equal to $\mathfrak{z}_V$. Set

$$\mathfrak{a}'_V = \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta , \qquad \mathfrak{r}'_V := \mathfrak{t} + \mathfrak{a}'_V .$$

Denote by R'_V the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\,\mathfrak{r}'_V$. The algebra $\mathfrak{a}'_V$ is in $\mathcal{C}'_\mathfrak{t}$ and has dimension $d - \dim \mathfrak{z}_V$. Then, by Lemma 2.3(ii), V is in $\overline{R'_V.\mathfrak{t}}$, whence the assertion since $\mathfrak{r}'_V$ is contained in $\mathfrak{r}^s$. □

Corollary 2.20. *Suppose that $\mathfrak{a}$ has Property* ($\mathbf{P}_1$). *Then $\mathfrak{a}$ has Property* ($\mathbf{P}$).

Proof. Let V be in X_R and s in $\mathfrak{t} \setminus \mathfrak{z}$ such that V is contained in $\mathfrak{r}^s$. As $\overline{\mathbf{T}.V}$ is a projective variety and $\mathbf{T}$ is a connected commutative group, $\mathbf{T}$ has a fixed point in $\overline{\mathbf{T}.V}$. Let V_∞ be such a point. Since all element of $\mathbf{T}.V$ is contained in $\mathfrak{r}^s$, so is V_∞. Then, by Lemma 2.19(iv), V_∞ is in $\overline{R^s.\mathfrak{t}}$. In particular, s is in V_∞. Let E a complement to V_∞ in $\mathfrak{r}$, invariant under $\mathbf{T}$. The map

$$\mathrm{Hom}_\Bbbk(V_\infty, E) \xrightarrow{\ \kappa\ } \mathrm{Gr}_d(\mathfrak{r}) , \qquad \varphi \longmapsto \kappa(\varphi) := \mathrm{span}(\{v + \varphi(v) \mid v \in V_\infty\})$$

is an isomorphism onto an open neighborhood Ω_E of V_∞ in $\mathrm{Gr}_d(\mathfrak{r})$. For φ in $\mathrm{Hom}_\Bbbk(V_\infty, E)$ such that $\kappa(\varphi)$ is in $\mathbf{T}.V$, $\varphi(s)$ is in $\mathfrak{a}^s$. Then, for some g in $\mathbf{T}$ and for some v in $\mathfrak{a}^s$, $s + v$ is in $g(V)$ and the semisimple component of $\mathrm{ad}\,(s + v)$ is different from 0 since s is not in $\mathfrak{z}$. Let x be in $\mathfrak{r}^s$ such that $\mathrm{ad}\,x$ is the semisimple component of $\mathrm{ad}\,(s + v)$. By Lemma 2.1(ii), for some k in R^s, $k(x)$ is in $\mathfrak{t}$. Then, by Corollary 2.14(ii), $kg(V)$ is in $\overline{R^{k(x)}.\mathfrak{t}}$. As $k(x)$ is not in $\mathfrak{z}$, $\mathfrak{a}^{k(x)}$ is an object of $\mathcal{C}'_\mathfrak{t}$ of dimension smaller than n. By hypothesis, $\mathfrak{a}^{k(x)}$ has Property ($\mathbf{P}$). Moreover, $kg(V)$ is contained in $\mathfrak{r}^s \cap \mathfrak{r}^{k(x)}$. Hence, by Property ($\mathbf{P}$) for $\mathfrak{a}^{k(x)}$, $kg(V)$ is in $\overline{R^s.\mathfrak{t}}$, whence V is in $\overline{R^s.\mathfrak{t}}$ since kg is in R^s. □

Proposition 2.21. *The objects of $\mathcal{C}'_\mathfrak{t}$ have Property* ($\mathbf{P}$).

Proof. Prove by induction on n that $\mathfrak{a}$ has Property ($\mathbf{P}$). By Lemma 2.8, it is true for $n = d^{\#}$. Suppose that it is true for the integers smaller than n. By Corollary 2.20, it remains to prove that $\mathfrak{a}$ has Property ($\mathbf{P}_1$).

Suppose that $\mathfrak{a}$ has not Property ($\mathbf{P}_1$). A contradiction is expected. For some fixed point V under $\mathbf{T}$ in X_R such that $V \cap \mathfrak{t} = \mathfrak{z}$, $r_V \neq |\mathcal{R}_V|$. By Lemma 2.19(ii), $r_V = |\mathcal{R}_V| - 1$. Then the orthogonal complement of $\mathcal{R}_V$ in $\mathfrak{t}$ is generated by $\mathfrak{z}$ and an element s in $\mathfrak{t} \setminus \mathfrak{z}$. In particular, V is contained in $\mathfrak{r}^s$. According to Lemma 2.5, for some ideal $\mathfrak{a}'$ of codimension 1 of $\mathfrak{a}$, normalized by $\mathfrak{t}$, $\mathfrak{a}^s$ is contained in $\mathfrak{a}'$. Denote by α the element of $\mathcal{R}$ such that

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^{\alpha}$$

and consider θ_α and Γ as in Subsection 2.5. Denote by Γ_V the set of elements of Γ whose image by the projection

$$\Gamma \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \ , \qquad (T_1, T', T, T_2) \longmapsto T$$

is equal to V. By Lemma 2.13(ii), Γ_V is not empty and it is invariant under $\mathbf{T}$ by Lemma 2.13(i). As it is a projective variety, it has a fixed point under $\mathbf{T}$. Denote by (V_1, V', V, W) such a point. As $\mathfrak{a}'$ has Property ($\mathbf{P}$), it has Property ($\mathbf{P}_1$) by Lemma 2.12. Hence $r_{V'} = |\mathcal{R}_{V'}|$ and $V' \neq V$ since $r_V \neq \mathcal{R}_V$. Then, by Lemma 2.13(iv),

$$V_1 = V \cap V' \quad \text{and} \quad W = V' + V.$$

As a result, $V' \cap \mathfrak{t} = V \cap \mathfrak{t} = \mathfrak{z}$ since $\mathcal{R}_{V'} \neq \mathcal{R}_V$ and V_1 has codimension 1 in V and V'. Then $V' = V$ by Corollary 2.18, whence a contradiction. □

The following corollary results from Proposition 2.21, Corollary 2.10 and Lemma 2.12.

Corollary 2.22. *Let V be in X_R.*

(i) *The space V is a commutative algebraic subalgebra of $\mathfrak{r}$ and for some subset Λ of $\mathcal{R}$, the biggest torus contained in V is conjugate to $\mathfrak{t}_\Lambda$ under R.*

(ii) *If V is a fixed point under R, then V is an ideal of $\mathfrak{r}$ and the elements of $\mathcal{R}_V$ are linearly independent.*

3. Solvable algebras and main varieties

Let $\mathfrak{t}$ be a vector space of positive dimension d and $\mathfrak{a}$ in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}, \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

In this section, we give some results on the singular locus of X_R. For $\mathfrak{a}'$ in $\mathcal{C}_\mathfrak{t}$, denote by $X_{R_{\mathfrak{t},\mathfrak{a}'},\mathfrak{n}}$ the subset of elements of $X_{R_{\mathfrak{t},\mathfrak{a}'}}$ contained in $\mathfrak{a}'$.

3.1. Subvarieties of X_R.

Denote by $\mathcal{P}_c(\mathcal{R})$ the set of complete subsets of $\mathcal{R}$ and for Λ in $\mathcal{P}_c(\mathcal{R})$ denote by X_{R_Λ} the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit $R_\Lambda.\mathfrak{t}$.

Proposition 3.1. *Let Z be an irreducible closed subset of X_R, invariant under R.*

(i) *For a well defined complete subset Λ of $\mathcal{R}$, all element of a dense open subset of Z is conjugate under R to the sum of $\mathfrak{t}_\Lambda$ and a subspace of $\mathfrak{a}$.*

(ii) *All element of Z is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$.*

(iii) *For some irreducible closed subset Z_Λ of X_{R_Λ}, invariant under R_Λ, $R.Z_\Lambda$ is dense in Z.*

Proof. (i) For Λ in $\mathcal{P}_c(\mathcal{R})$, let Y_Λ be the subset of elements V of Z such that $\pi(V) = \mathfrak{t}_\Lambda$. Since Z is invariant under R, so is Y_Λ. Moreover, by Corollary 2.22(i),

$$\overline{Y_\Lambda} \subset Y_\Lambda \cup \bigcup_{\substack{\Lambda' \in \mathcal{P}_c(\mathcal{R}) \\ \Lambda' \supsetneq \Lambda}} Y_{\Lambda'}.$$

According to Corollary 2.22(i), Z is the union of $Y_\Lambda, \Lambda \in \mathcal{P}_c(\mathcal{R})$. As a result, since $\mathcal{R}$ is finite and Z is irreducible, for a well defined complete subset Λ of $\mathcal{R}$, Y_Λ contains a dense open subset of Z. By Lemma 2.1(v), all element of Y_Λ is conjugate under R to the sum of $\mathfrak{t}_\Lambda$ and a subspace of $\mathfrak{a}$.

(ii) By (i), for all V in a dense subset of Z, V is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$, whence the assertion.

(iii) Let Z_* be the subset of elements of Z, containing $\mathfrak{t}_\Lambda$. Denote by s an element of $\mathfrak{t}_\Lambda$ such that $\alpha(s) \neq 0$ for all α in $\mathcal{R} \setminus \Lambda$. By Lemma 2.6(i),

$$\mathfrak{r}^s = \mathfrak{t} \oplus \mathfrak{a}_\Lambda .$$

Hence Z_* is contained in X_{R_Λ} by Proposition 2.21. Moreover, Z_* is invariant under R_Λ since Z is invariant under R. By (i), $R.Z_*$ is dense in Z. So, for some irreducible component Z_Λ of Z_*, $R.Z_\Lambda$ is dense in Z. Moreover, Z_Λ is invariant under R_Λ since so is Z_*. □

For Λ in $\mathcal{P}_c(\mathcal{R})$, denote by $\mathfrak{t}_\Lambda^\#$ a complement to $\mathfrak{t}_\Lambda$ in $\mathfrak{t}$ and set:

$$\mathfrak{r}_\Lambda^\# := \mathfrak{t}_\Lambda^\# + \mathfrak{a}_\Lambda .$$

Let $R_\Lambda^\#$ be the adjoint group of $\mathfrak{r}_\Lambda^\#$ and $A_\Lambda^\#$ the connected closed subgroup of $R_\Lambda^\#$ whose Lie algebra is $\mathrm{ad}\,\mathfrak{a}_\Lambda$.

Lemma 3.2. *Let Λ be in $\mathcal{P}_c(\mathcal{R})$, nonempty and strictly contained in $\mathcal{R}$.*

(i) *The tori $\mathfrak{t}_\Lambda$ and $\mathfrak{t}_\Lambda^\#$ have positive dimension and $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}_\Lambda^\#}$. Moreover,*

$$\dim \mathfrak{a}_\Lambda - \dim \mathfrak{t}_\Lambda^\# \leqslant \dim \mathfrak{a} - \dim \mathfrak{t}.$$

(ii) *The map $V \mapsto V \oplus \mathfrak{t}_\Lambda$ is an isomorphism from $X_{R_\Lambda^\#}$ onto X_{R_Λ}.*

Proof. Since Λ is a complete subset of $\mathcal{R}$ strictly contained in $\mathcal{R}$, $\mathfrak{t}_\Lambda$ has positive dimension and since Λ is not empty, $\mathfrak{t}_\Lambda$ is strictly contained in $\mathfrak{t}$. By definition, Λ is the set of weights of $\mathfrak{t}$ in $\mathfrak{a}_\Lambda$ so that $\mathfrak{a}_\Lambda$ is in $\mathcal{C}'_{\mathfrak{t}}$. Then $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}_\Lambda^\#}$ and Assertion (ii) results from Corollary 2.2.

By Lemma 2.1,(i) and (iv), $\mathcal{R}$ generates $\mathfrak{t}^*$. Hence

$$|\Lambda| + \dim \mathfrak{t}_\Lambda \leqslant |\mathcal{R}|.$$

By Condition (2) of Section 2, $\mathfrak{a}$ has dimension $|\mathcal{R}|$ and $\mathfrak{a}_\Lambda$ has dimension $|\Lambda|$. As a result,

$$\dim \mathfrak{a} - \dim \mathfrak{t} = |\mathcal{R}| - \dim \mathfrak{t}_\Lambda - \dim \mathfrak{t}_\Lambda^\# \geqslant \dim \mathfrak{a}_\Lambda - \dim \mathfrak{t}_\Lambda^\# .$$

□

3.2. **Smooth points of X_R and commutators.** Denote by $\mathfrak{t}_{\mathrm{reg}}$ the complement in $\mathfrak{t}$ to the union of $\mathfrak{t}_\alpha, \alpha \in \mathcal{R}$ and $\mathfrak{r}_{\mathrm{reg}}$ the set of elements x of $\mathfrak{r}$ such that $\mathfrak{r}^x$ has minimal dimension.

Lemma 3.3. (i) *The set $\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{t}$, contained in $\mathfrak{r}_{\mathrm{reg}}$. Moreover, $R.\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{r}$.*

(ii) *For all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ is in X_R.*

(iii) *The set $\mathfrak{r}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{r}$.*

Proof. (i) By definition, $\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{t}$. According to Lemma 2.6(i), for x in $\mathfrak{t}_{\mathrm{reg}}$, $\mathfrak{r}^x = \mathfrak{t}$. Then $R.x = A.x = x + \mathfrak{a}$ since $A.x$ is a closed subset of $x + \mathfrak{a}$ of dimension $\dim \mathfrak{a}$. As a result, $R.\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{a}$ is a dense open subset of $\mathfrak{r}$. Hence $R.\mathfrak{t}_{\mathrm{reg}}$ is contained in $\mathfrak{r}_{\mathrm{reg}}$ since $\mathfrak{r}^x$ is conjugate to $\mathfrak{t}$ for all x in $R.\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{r}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{r}$.

(ii) By (i), for all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ has dimension d, whence a regular map

$$\mathfrak{r}_{\mathrm{reg}} \xrightarrow{\ \theta\ } \mathrm{Gr}_d(\mathfrak{r}) , \qquad x \longmapsto \mathfrak{r}^x .$$

As a result, by (i), for all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ is in X_R.

(iii) Suppose that $\mathfrak{r}_{\mathrm{reg}}$ is not a big open subset of $\mathfrak{r}$. A contradiction is expected. Let Σ be an irreducible component of codimension 1 of $\mathfrak{r} \setminus \mathfrak{r}_{\mathrm{reg}}$. Since $\Sigma \cap A.\mathfrak{t}_{\mathrm{reg}}$ is empty, $\pi(\Sigma)$ is contained in $\mathfrak{t}_\alpha$ for some α in $\mathfrak{r}$. Then $\Sigma = \mathfrak{t}_\alpha + \mathfrak{a}$ since Σ has codimension 1 in $\mathfrak{r}$. By Condition (3) of Section 2, for some s in $\mathfrak{t}_\alpha$, $\gamma(s) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $\mathfrak{r}^{s+x_\alpha} = \mathfrak{t}_\alpha + \mathfrak{a}^\alpha$ so that $s + x_\alpha$ is in $\mathfrak{r}_{\rm reg}$ by (i) and Condition (2) of Section 2, whence the contradiction. □

Denote by X'_R the image of θ.

Proposition 3.4. (i) *The complement to $R.\mathfrak{t}$ in X_R is equidimensional of dimension* $\dim \mathfrak{a} - 1$.

(ii) *The set X'_R is a smooth open subset of X_R, containing $R.\mathfrak{t}$.*

Proof. (i) As R is solvable and $R.\mathfrak{t}$ is dense in X_R, $R.\mathfrak{t}$ is an affine open subset of X_R. So, by [EGAIV, Corollaire 21.12.7], $X_R \setminus R.\mathfrak{t}$ is equidimensional of dimension $\dim \mathfrak{a} - 1$ since X_R has dimension $\dim \mathfrak{a}$.

(ii) By definition, $\mathcal{E}$ is the subvariety of elements (V, x) of $X_R \times \mathfrak{r}$ such that x is in V. Let Γ be the image of the graph of θ by the isomorphism

$$\mathfrak{r} \times \mathrm{Gr}_d(\mathfrak{r}) \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \times \mathfrak{r} \ , \qquad (x, V) \longmapsto (V, x).$$

Then Γ is the intersection of $\mathcal{E}$ and $X_R \times \mathfrak{r}_{\rm reg}$. Since Γ is isomorphic to $\mathfrak{r}_{\rm reg}$, Γ is a smooth open subset of $\mathcal{E}$ whose image by the bundle projection is X'_R. As a result, X'_R is a smooth open subset of X_R by [MA86, Ch. 8, Theorem 23.7]. □

For α in $\mathcal{R}$, set $V_\alpha := \mathfrak{t}_\alpha \oplus \mathfrak{a}^\alpha$ and denote by θ_α the map

$$\Bbbk \xrightarrow{\theta_\alpha} \mathrm{Gr}_d(\mathfrak{r}) \ , \qquad z \longmapsto \exp(z \operatorname{ad} x_\alpha)(\mathfrak{t}),$$

By Condition (2) of Section 2, V_α has dimension d.

Lemma 3.5. *Let α be in $\mathcal{R}$. Set $X_{R,\alpha} := \overline{A.V_\alpha}$.*

(i) *The map θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$ such that $\theta_\alpha(\infty) = V_\alpha$.*

(ii) *The variety $X_{R,\alpha}$ has dimension* $\dim \mathfrak{a} - 1$ *and it is an irreducible component of $X_R \setminus R.\mathfrak{t}$.*

(iii) *The intersection $X_{R,\alpha} \cap X'_R$ is not empty.*

Proof. (i) Let h_α be in $\mathfrak{t}$ such that $\alpha(h_\alpha) = 1$. Since X_R is a projective variety, the map θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$ by [Sh94, Ch. 6, Theorem 6.1]. For z in $\Bbbk$,

$$\theta_\alpha(z) = \mathfrak{t}_\alpha \oplus \Bbbk(h_\alpha - z x_\alpha).$$

Hence $\theta_\alpha(\infty) = V_\alpha$.

(ii) By (i), $X_{R,\alpha}$ is contained in X_R and its elements are contained in $\mathfrak{t}_\alpha \oplus \mathfrak{a}$ so that $X_{R,\alpha}$ is contained in $X_R \setminus R.\mathfrak{t}$. By Condition (3) of Section 2, for γ in $\mathcal{R} \setminus \{\alpha\}$ and v in $\mathfrak{a}^\gamma$, $[\mathfrak{t}_\alpha, v] = \Bbbk v$ so that no element of $\mathfrak{a}^\gamma$ normalizes V_α. As a result, the normalizer of V_α in $\mathfrak{r}$ is equal to $\mathfrak{t} + \mathfrak{a}^\alpha$ so that $X_{R,\alpha}$ has dimension $\dim \mathfrak{a} - 1$. Hence $X_{R,\alpha}$ is an irreducible component $X_R \setminus R.\mathfrak{t}$.

(iii) According to Condition (3) of Section 2, for some s in $\mathfrak{t}_\alpha$, $\gamma(s) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $V_\alpha = \mathfrak{r}^{s+x_\alpha}$ so that $s + x_\alpha$ is in $\mathfrak{r}_{\rm reg}$, whence the assertion. □

3.3. **On the singular locus of X_R.** In this subsection we suppose $\dim \mathfrak{a} > d$ and we fix an ideal $\mathfrak{a}'$ of codimension 1 in $\mathfrak{a}$, normalized by $\mathfrak{t}$ and such that $\mathfrak{a}'$ is in $\mathcal{C}_\mathfrak{t}$. For example, all ideal of $\mathfrak{r}$ of dimension $\dim \mathfrak{a} - 1$, contained in $\mathfrak{a}$ and containing a fixed point under R in X_R is in $\mathcal{C}_\mathfrak{t}$ by Corollary 2.22(ii). Set:

$$\mathfrak{r}' := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}'} \qquad \pi' := \pi_{\mathfrak{t},\mathfrak{a}'}, \qquad R' := R_{\mathfrak{t},\mathfrak{a}'}, \qquad A' := A_{\mathfrak{t},\mathfrak{a}'}, \qquad \mathcal{R}' := \mathcal{R}_{\mathfrak{t},\mathfrak{a}'}.$$

Let α be in $\mathcal{R}$ such that

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^\alpha$$

and Γ as in Subsection 2.5. Denote by $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ the restrictions to Γ of the first, second, third, fourth projections. Let Z be an irreducible component of $X_{R,\mathfrak{n}}$. According to Lemma 2.13(ii), for some irreducible component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2 and by T_1 the image of T by the projection $\varpi_1\times\varpi_4$. Then Z' and T_1 are irreducible closed subsets of $\mathrm{Gr}_d(\mathfrak{r})$ and $\mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ respectively. Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Then T_0 is a closed subset of T. If $T_0 = T$, $Z' = Z$ and Z is contained in $X_{R',\mathrm{n}}$. Otherwise, $O := T \setminus T_0$ is a dense open subset of T. According to Lemma 2.13(iv), for all (V_1, V', V, W) in O, $V_1 = V' \cap V$ and $V' + V = W$. Denote by O_1 an open subset of T_1, contained and dense in $\varpi_1\times\varpi_4(O)$.

Let (V_1, W) be in O_1. Denote by E a complement to V_1 in $\mathfrak{r}$ and by E' a complement to W in $\mathfrak{r}$ contained in E. Let κ be the map

$$\mathrm{Hom}_{\Bbbk}(V_1, W\cap E) \times \mathrm{Hom}_{\Bbbk}(W, E') \xrightarrow{\ \kappa\ } \mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) \ ,$$
$$(\varphi, \psi) \longmapsto (\mathrm{span}(\{v + \varphi(v) + \psi(v) + \psi\circ\varphi(v) \mid v \in V_1\}), \mathrm{span}(\{v + \psi(v) \mid v \in W\})).$$

Then κ is an isomorphism from its source to an open neighborhood of (V_1, W) in the subvariety of elements (W_1, W_2) of $\mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ such that W_1 is contained in W_2. Denote by Ω the inverse image by κ of the intersection of T_1 and the image of κ. Let (e_1, e_2) be a basis of $W \cap E$ and let κ_* be the map

$$\Omega \times (\Bbbk^2 \setminus \{(0, 0)\}) \xrightarrow{\ \kappa_*\ } \mathrm{Gr}_d(\mathfrak{r}) \ ,$$
$$(\varphi, \psi, x_1, x_2) \longmapsto \mathrm{span}(\{v + \varphi(v) + \psi(v) + \psi\circ\varphi(v) \mid v \in V_1\} \cup \{x_1(e_1 + \psi(e_1)) + x_2(e_2 + \psi(e_2))\}).$$

Lemma 3.6. *Suppose that O is not empty. Denote by $\widetilde{\Omega}$ the image of κ_* and $\tilde{Z}$ the closure of $\widetilde{\Omega}$ in $\mathrm{Gr}_d(\mathfrak{r})$.*

(i) *The intersections $\widetilde{\Omega} \cap Z'$ and $\widetilde{\Omega} \cap Z$ are dense in Z' and Z respectively. In particular Z' and Z are contained in $\tilde{Z}$.*

(ii) *For V in $\widetilde{\Omega}$, there exists (V', V'') in $Z' \times Z$ such that*

$$V' \cap V'' \subset V, \qquad V \subset V' + V'', \qquad (V' \cap V'', V' + V'') \in \kappa(\Omega).$$

(iii) *Let F' be the fiber of κ_* at some element V of $\kappa_*(\Omega)$. Denote by F the subset of elements (φ, ψ) of Ω such that V contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Then $F' = F \times \Bbbk^*(x_1, x_2)$ for some (x_1, x_2) in $\Bbbk^2 \setminus \{(0, 0)\}$.*

(iv) *The varieties $\tilde{Z}$ and Z have dimension at most* $\dim Z' + 1$.

Proof. (i) Since T is irreducible so are T_1 and Ω. Hence $\tilde{Z}$ is irreducible. For some (V', V) in $Z' \times Z$, V_1 is contained in V' and V and V' and V are contained in W. Since $\kappa(\Omega)$ is an open neighbourhood of (V_1, W) in T_1,

$$\varpi_2(\varpi_1\times\varpi_4^{-1}(\kappa(\Omega)) \cap T) \quad \text{and} \quad \varpi_3(\varpi_1\times\varpi_4^{-1}(\kappa(\Omega)) \cap T)$$

are dense subsets of Z' and Z respectively. For all (φ, ψ) in Ω, all element of $\varpi_2(\varpi_1\times\varpi_4^{-1}(\kappa(\varphi, \psi)) \cap T)$ contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Hence all element of $\varpi_2(\varpi_1\times\varpi_4^{-1}(\kappa(\Omega)) \cap T)$ is in the image of κ_*. As a result, $\widetilde{\Omega} \cap Z'$ is dense in Z' and Z' is contained in $\tilde{Z}$. In the same way, $\widetilde{\Omega} \cap Z$ is dense in Z and Z is contained in $\tilde{Z}$.

(ii) According to Lemma 2.13(iv), for all (V_1', V', V, W') in O, $V_1' = V' \cap V$ and $W' = V' + V$. By definition, $\kappa(\Omega)$ is contained in $\varpi_1\times\varpi_4(O)$ and for V in $\widetilde{\Omega}$, $V_1' \subset V$ and $V \subset W'$ for some (V_1', W') in $\kappa(\Omega)$, whence the assertion.

(iii) For (φ, ψ) in F and for (x_1, x_2) in $\Bbbk^2 \setminus \{(0, 0)\}$ such that

$$V = \kappa_*(\varphi, \psi, x_1, x_2),$$

the subset of elements (y_1, y_2) of $\Bbbk^2$ such that $(\varphi, \psi, y_1, y_2)$ is in F' is equal to $\Bbbk^*.(x_1, x_2)$. Moreover, for all $(\varphi, \psi, y_1, y_2)$ in F', (φ, ψ) is in F, whence the assertion.

(iv) In (iii), we can choose V such that F' has minimal dimension so that

$$\dim \tilde{Z} = \dim \Omega + 2 - (\dim F + 1) = \dim \Omega - \dim F + 1.$$

By (ii), for some V' in Z', for all (φ, ψ) in F, V' contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. So, again by (iii) and (ii),

$$\dim Z' \geqslant \dim \Omega - \dim F,$$

whence $\dim \tilde{Z} \leqslant \dim Z' + 1$ and $\dim Z \leqslant \dim Z' + 1$ since Z is contained in $\tilde{Z}$ by (i). □

Proposition 3.7. *The variety $X_{R,\mathrm{n}}$ has dimension at most $n - d$.*

Proof. Prove this by induction on n. According to Lemma 2.3(ii), it is true for $n - d = 0$. Suppose that $n - d$ is positive and that it is true for all integer smaller than $n - d$. In particular, $X_{R',\mathrm{n}}$ has dimension at most $n - d - 1$. Let Z be an irreducible component of $X_{R,\mathrm{n}}$. According to Lemma 2.13(ii), for some irreducible component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2. Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Consider the following cases:

(a) $T_0 = T$,
(b) $T_0 \neq T$ and Z' is contained in $X_{R',\mathrm{n}}$,
(c) Z' is not contained in $X_{R',\mathrm{n}}$.

(a) In this case, $Z' = Z$ and $\dim Z \leqslant n - d - 1$ by induction hypothesis.

(b) By induction hypothesis, $\dim Z' \leqslant n - d - 1$ and by Lemma 3.6(iv), $\dim Z \leqslant \dim Z' + 1$, whence $\dim Z \leqslant n - d$.

(c) In this case, $T_0 \neq T$, whence $\dim Z \leqslant \dim Z' + 1$ by Lemma 3.6(iv). Since Z is an irreducible component of $X_{R,\mathrm{n}}$, Z is invariant under R. By Lemma 2.13(i), ϖ_2 and ϖ_3 are equivariant under the action of R' in Γ so that T and Z' are invariant under R'. For all (V_1, V', V, W) in $T \setminus T_0$, $V_1 = V' \cap V$. Hence all element of a dense open subset of Z' contains a subspace of dimension $d - 1$ of $\mathfrak{a}'$. Then, by Proposition 3.1, for some complete subset Λ of $\mathcal{R}'$ such that $\mathfrak{t}_\Lambda$ has dimension 1 and for some closed subset Z_Λ of X_{R_Λ}, $R'.Z_\Lambda$ is dense in Z' so that

$$\dim Z' \leqslant \dim Z_\Lambda + \dim \mathfrak{a}' - \dim \mathfrak{a}_\Lambda.$$

If $\dim \mathfrak{a}_\Lambda - \dim \mathfrak{t} + 1 = n - d$, then $\Lambda = \mathcal{R}'$. In this case, since $\mathfrak{a}'$ is in $\mathcal{C}_\mathfrak{t}$, Λ generates $\mathfrak{t}^*$. As $\mathfrak{t}_\Lambda$ has dimension 1, it is impossible. As a result,

$$\dim Z_\Lambda \leqslant \dim \mathfrak{a}_\Lambda - \dim \mathfrak{t} + 1 \quad \text{and} \quad \dim Z' \leqslant n - d$$

by Lemma 3.2 and induction hypothesis for $\mathfrak{a}_\Lambda$. Then $\dim Z \leqslant n - d + 1$. According to Lemma 3.6,(i) and (iv), $\tilde{Z}$ is an irreducible variety of dimension at most $\dim Z' + 1$, containing Z' and Z. If $\dim Z' = n - d$ and $\dim Z = n - d + 1$, then $Z = \tilde{Z}$. In particular, Z' is contained in Z. It is impossible since all element of Z is contained in $\mathfrak{a}$. As a result, $\dim Z \leqslant n - d$, whence the proposition. □

Corollary 3.8. (i) *The irreducible components of $X_R \setminus R.\mathfrak{t}$ are the $X_{R,\alpha}$, $\alpha \in \mathcal{R}$.*

(ii) *The set X'_R is a smooth big open subset of X_R, containing $R.\mathfrak{t}$.*

Proof. According to Proposition 3.4(ii) and Lemma 3.5(iii), Assertion (ii) results from Assertion (i). Prove Assertion (i) by induction on $n = \dim \mathfrak{a}$. For $n = 1$, $d = 1$ by Lemma 2.1,(i) and (iv) so that X_R is the union of $R.\mathfrak{t}$ and $\mathfrak{a}^\alpha$, whence Assertion (i) in this case. Suppose $n \geqslant 2$ and the assertion true for the integers smaller than n. By Lemma 2.1(i), Condition (2) and Condition (3) of Section 2, $d \geqslant 2$. According to Lemma 3.5(ii), for all α in $\mathcal{R}$, $X_{R,\alpha}$ is an irreducible component of $X_R \setminus R.\mathfrak{t}$. Let Z be an irreducible component of $X_R \setminus R.\mathfrak{t}$. By Proposition 3.4(i), Z has dimension $n - 1$. So, by Proposition 3.7, Z is not contained in $X_{R,\mathrm{n}}$. Moreover,

Z is invariant under R. Then, by Proposition 3.1, for some complete subset Λ of $\mathcal{R}$, strictly contained in $\mathcal{R}$ and for some irreducible closed subset Z_Λ of X_{R_Λ}, invariant under R_Λ, $R.Z_\Lambda$ is dense in Z. By Lemma 3.2, $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}_\Lambda^\#}$ and Z_Λ is the image of a closed subset Z'_Λ of $X_{R_\Lambda^\#}$, invariant by $R_\Lambda^\#$, by the map $V \mapsto V \oplus \mathfrak{t}_\Lambda$. Since Z_Λ is contained in Z, $Z'_\Lambda \cap R_\Lambda^\#.\mathfrak{t}_\Lambda^\#$ is empty. As Λ is strictly contained in $\mathcal{R}$, $\dim \mathfrak{a}_\Lambda$ is smaller than n. So, by induction hypothesis, for some α in Λ, Z'_Λ is contained in $X_{R_\Lambda^\#,\alpha}$. As a result, Z_Λ and Z are contained in $X_{R,\alpha}$, whence $Z = X_{R,\alpha}$ since Z is an irreducible component of $X_R \setminus R.\mathfrak{t}$. □

4. Normality for solvable Lie algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d and $\mathfrak{a}$ in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is normal and Cohen-Macaulay.

4.1. **The case** $\dim \mathfrak{a} = \dim \mathfrak{t}$**.** By Condition (2) of Section 2, $\mathcal{R}$ has d elements $\beta_1, \dots, \beta_d$ linearly independent. Denote by $t_1, \dots, t_d$ the dual basis in $\mathfrak{t}$. For $i = 1, \dots, d$, let v_i be a generator of $\mathfrak{a}^{\beta_i}$.

Lemma 4.1. *If* $\dim \mathfrak{a} = \dim \mathfrak{t}$ *then* X_R *is a smooth variety. Moreover, for all* $(z_1, \dots, z_d)$ *in* $\Bbbk^d$*, the subspace generated by* $v_1 + z_1 t_1, \dots, v_d + z_d t_d$ *is in* X_R.

Proof. According to Lemma 2.3, $\mathfrak{a}$ is in in X_R and the map

$$\Bbbk^d \longrightarrow X_R \ , \qquad (z_1, \dots, z_d) \longmapsto \operatorname{span}(\{v_1 + z_1 t_1, \dots, v_d + z_d x_d\})$$

is an isomorphism onto an open neighborhood of $\mathfrak{a}$ in X_R. Hence $\mathfrak{a}$ is a smooth point of X_R. By Corollary 2.22, R has only one fixed point $\mathfrak{a}$ in X_R. Since for all V in X_R, R has a fixed point in $\overline{R.V}$ and $X_{R\mathrm{sm}}$ is an open subset of X_R, invariant under R, $X_R = X_{R\mathrm{sm}}$. □

4.2. **Cohen-Macaulayness property for some algebras.** Let A_* be an integral domain and a local commutative $\Bbbk$-algebra with maximal ideal $\mathfrak{m}$ and $u_1, \dots, u_s$ a regular sequence in A_* of elements of $\mathfrak{m}$. Let $T_1, \dots, T_s$ be indeterminates. Set $B_s := A_*[T_1, \dots, T_s]$ and denote by P_s and P'_s the ideals of B_s generated by the sequences $u_j T_k - u_k T_j$, $1 \leqslant j, k \leqslant s$ and $u_j T_1 - u_1 T_j$, $1 \leqslant j \leqslant s$ respectively.

Lemma 4.2. *The ideal* P_s *is a prime ideal of* B_s.

Proof. For $s = 1$, $P_s = \{0\}$. Suppose $s \geqslant 2$. Let $\tilde{P}$ be the ideal of $B_s[T_1^{-1}]$ generated by P_s. For $1 \leqslant j, k \leqslant s$,

$$T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1).$$

Hence $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by P'_s. Setting $S_j := T_j/T_1$ for $j = 2, \dots, s$, denote by C the polynomial algebra $A_*[S_2, \dots, S_s]$ over A_* so that $B_s[T_1^{-1}] = C[T_1, T_1^{-1}]$ and $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by $u_j - u_1 S_j$, $j = 2, \dots, s$.

Claim 4.3. Let Q be the ideal of C generated by $u_j - u_1 S_j$, $j = 2, \dots, s$. Then Q is prime.

Proof. [Proof of Claim 4.3] Let $\tilde{Q}$ be the ideal of $C[u_1^{-1}]$ generated by Q. Then $\tilde{Q}$ is prime since it is generated by $u_j u_1^{-1} - S_j$, $j = 2, \dots, s$. As a result, for p and q in C such that pq is in Q, for some nonnegative integer m, $u_1^m p$ or $u_1^m q$ is in Q. So it remains to prove that for p in C, p is in Q if so is $u_1 p$.

Let p be in C such that $u_1 p$ is in Q. For some $q_2, \dots, q_s$ in C,

$$u_1 p = \sum_{j=2}^{s} q_j (u_j - u_1 S_j) \quad \text{whence} \quad \sum_{j=1}^{s} q_j u_j = 0 \quad \text{with} \quad q_1 := -(p + \sum_{j=2}^{s} q_j S_j).$$

By hypothesis, the sequence $u_1, \ldots, u_s$ is regular in C. So for some sequence $q_{j,k}, 1 \leqslant j, k \leqslant s$ in C such that $q_{j,k} = -q_{k,j}$,

$$q_j = \sum_{k=1}^{s} q_{j,k} u_k$$

for $j = 1, \ldots, s$. As a result,

$$\begin{aligned} u_1 p = &\quad \textstyle\sum_{j=2}^{s} \sum_{k=1}^{s} q_{j,k} u_k (u_j - u_1 S_j) \\ = &\quad \textstyle\sum_{j=2}^{s} q_{j,1} u_j u_1 - \sum_{j=2}^{s} \sum_{k=1}^{s} q_{j,k} u_k u_1 S_j \\ = &\quad u_1 (\textstyle\sum_{j=2}^{s} q_{j,1} (u_j - u_1 S_j) + \sum_{2 \leqslant j < k \leqslant s} q_{j,k} (u_j S_k - u_k S_j)). \end{aligned}$$

For $2 \leqslant j, k \leqslant s$,

$$u_j S_k - u_k S_j = (u_j - u_1 S_j) S_k - (u_k - u_1 S_k) S_j \in Q,$$

whence the claim. □

By the claim, $\tilde{P}$ is a prime ideal of $B_s[T_1^{-1}]$ since it is generated by Q. As a result for p and q in B_s such that pq is in P_s, for some nonnegative integer m, $T_1^m p$ or $T_1^m q$ is in P'_s since $T_1 P_s$ is contained in P'_s by the equality

$$T_1 (u_j T_k - u_k T_j) = T_k (u_j T_1 - u_1 T_j) + T_j (u_1 T_k - u_k T_1)$$

for $1 \leqslant i, j \leqslant s$. So it remains to prove that for p in B_s, p is in P_s if $T_1 p$ is in P'_s.

Let p be in B_s such that $T_1 p$ is in P'_s. For some $r_2, \ldots, r_s$ in B_s,

$$T_1 p = \sum_{j=2}^{s} r_j (u_j T_1 - u_1 T_j).$$

For $j = 2, \ldots, s$, r_j has an expansion

$$r_j = r_{j,0} + T_1 r_{j,1}$$

with $r_{j,0}$ and $r_{j,1}$ in $B'_s := A_*[T_2, \ldots, T_s]$ and B_s respectively. Set:

$$p' := p - \sum_{j=2}^{s} r_{j,1} (u_j T_1 - u_1 T_j).$$

Then

$$T_1 p' = \sum_{j=2}^{s} r_{j,0} (u_j T_1 - u_1 T_j)$$

so that the element

$$\sum_{j=2}^{s} r_{j,0} u_1 T_j \in B'_s$$

is divisible by T_1 in B_s, whence

$$\sum_{j=2}^{s} r_{j,0} T_j = 0.$$

As $T_2, \ldots, T_s$ is a regular sequence in B_s, for some sequence $r_{j,k,0}, 2 \leqslant j, k \leqslant s$ in B_s such that $r_{j,k,0} = -r_{k,j,0}$ for all (j, k),

$$r_{j,0} = \sum_{k=2}^{s} r_{j,k,0} T_k$$

for $j = 2, \ldots, s$. Then

$$T_1 p' = \sum_{2 \leqslant j,k \leqslant s} r_{j,k,0} T_k(u_j T_1 - u_1 T_j) = T_1 \sum_{2 \leqslant j<k \leqslant s} r_{j,k,0}(T_k u_j - T_j u_k).$$

As a result p' and p are in P_s, whence the lemma. □

Denote by P''_s the ideal of B_s generated by P_{s-1} and $u_s T_1 - u_1 T_s$. Let $\mathfrak{B}_s$ and $\mathfrak{B}'_s$ be the quotients of B_s by P_s and P''_s respectively. The restrictions to A_* of the quotient morphisms $B_s \longrightarrow \mathfrak{B}'_s$ and $B_s \longrightarrow \mathfrak{B}_s$ are embeddings. For $j = 1, \ldots, s$, denote again by T_j its images in $\mathfrak{B}'_s$ and $\mathfrak{B}_s$ by these morphisms.

Lemma 4.4. *Denote by $\overline{P_s}$ the image in $\mathfrak{B}'_s$ of P_s by the quotient morphism.*

(i) *The intersection of $\overline{P_s}$ and $T_1\mathfrak{B}'_s$ is equal to* $\{0\}$.

(ii) *The $\mathfrak{B}'_s$-modules $T_1\mathfrak{B}'_s$ and $\mathfrak{B}_s$ are isomorphic.*

Proof. Let a be in B_s such that $T_1 a$ is in P_s. According to Lemma 4.2, P_s is a prime ideal of B_s. Hence a is in P_s since T_1 is not in P_s. Moreover, for $j = 1, \ldots, s$,

$$T_1(u_j T_s - u_s T_j) = T_s(u_j T_1 - u_1 T_j) + T_j(u_1 T_s - u_s T_1).$$

Hence $T_1 P_s$ is contained in P''_s. As a result, $\overline{P_s}$ is the kernel of the endomorphism $a \mapsto T_1 a$ of $\mathfrak{B}'_s$ and the intersection of $\overline{P_s}$ and $T_1\mathfrak{B}'_s$ is equal to $\{0\}$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, the endomorphism $a \mapsto T_1 a$ defines through the quotient an isomorphism

$$\mathfrak{B}_s \longrightarrow T_1\mathfrak{B}'_s$$

of $\mathfrak{B}'_s$-modules. □

Let Q_s be the ideal of the polynomial algebra $A_*[T_2, \ldots, T_s]$ generated by the sequence $u_i T_k - u_k T_i$, $2 \leqslant i, k \leqslant s$ and denote by $\mathfrak{B}^{\#}_s$ the quotient of $A_*[T_2, \ldots, T_s]$ by Q_s.

Lemma 4.5. (i) *The quotient of the algebra $\mathfrak{B}_s/T_1\mathfrak{B}_s$ by the ideal generated by u_1 is equal to the quotient of $\mathfrak{B}^{\#}_s$ by the ideal generated by u_1.*

(ii) *The canonical map $A_* \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s$ is an embedding.*

(iii) *The ideal of $\mathfrak{B}_s/T_1\mathfrak{B}_s$ generated by u_1 is isomorphic to A_**

Proof. Denote by Q'_s the ideal of B_s generated by P_s and T_1.

(i) As the ideal of B_s generated by Q'_s and u_1 is equal to the ideal generated by u_1, T_1 and Q_s, $\mathfrak{B}^{\#}_s/u_1\mathfrak{B}^{\#}_s$ is equal to the quotient of $\mathfrak{B}_s/T_1\mathfrak{B}_s$ by the ideal generated by u_1.

(ii) Since the intersection of A_* and Q'_s is equal to $\{0\}$, the canonical map $A_* \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s$ is an embedding.

(iii) For $k = 2, \ldots, s$, $u_1 T_k$ is in Q'_s. Hence $u_1 B_s$ is contained in the sum of $u_1 A_*$ and Q'_s. As a result, $u_1 A_*$ is equal to $u_1\mathfrak{B}_s/T_1\mathfrak{B}_s$ by (ii), whence the assertion since A_* is an integral domain. □

Proposition 4.6. *Suppose that A_* is Cohen-Macaulay.*

(i) *The algebra $\mathfrak{B}_s$ is an integral domain and a Cohen-Macaulay algebra of dimension* $\dim A_* + 1$.

(ii) *For $a_1, \ldots, a_m$ regular sequence in A_* of elements of $\mathfrak{m}$ an for $\mathfrak{p}$ prime ideal of $\mathfrak{B}_s$ containing it, $a_1, \ldots, a_m$ is a regular sequence in the localization of $\mathfrak{B}_s$ at $\mathfrak{p}$.*

Proof. (i) Prove the assertion by induction on s. As $\mathfrak{B}_1$ is the polynomial algebra $A_*[T_1]$, the assertion is true for $s = 1$ since A_* an integral domain and a Cohen-Macaulay algebra. Suppose the assertion true for $s - 1$. By induction hypothesis, $\mathfrak{B}_{s-1}[T_s]$ is an integral domain and a Cohen-Macaulay algebra as a polynomial algebra over $\mathfrak{B}_{s-1}$ and its dimension is equal to $\dim A_* + 2$. As a result, $\mathfrak{B}'_s$ is Cohen-Macaulay of dimension $\dim A_* + 1$ as the quotient of the integral domain and a Cohen-Macaulay algebra $\mathfrak{B}_{s-1}[T_s]$ by the ideal generated by $T_s u_1 - T_1 u_s$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, $\mathfrak{B}_s$ has dimension at most $\dim A_* + 1$. By Lemma 4.2, $\mathfrak{B}_s$ is an integral domain so that $\mathfrak{B}_s/T_1\mathfrak{B}_s$ has dimension at most $\dim A_*$.

By induction hypothesis again, $\mathfrak{B}_s^\#$ is an integral domain and a Cohen- Macaulay algebra of dimension $\dim A_* + 1$. Hence $\mathfrak{B}_s^\#/u_1\mathfrak{B}_s^\#$ is Cohen-Macaulay of dimension $\dim A_*$. According to Lemma 4.5, we have a short exact sequence

$$0 \longrightarrow A_* \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s \longrightarrow \mathfrak{B}_s^\#/u_1\mathfrak{B}_s^\# \longrightarrow 0 \ .$$

Hence the algebra $\mathfrak{B}_s/T_1\mathfrak{B}_s$ is Cohen-Macaulay of dimension $\dim A_*$ since A_* and $\mathfrak{B}_s^\#/u_1\mathfrak{B}_s^\#$ are Cohen-Macaulay of dimension $\dim A_*$ and $\mathfrak{B}_s/T_1\mathfrak{B}_s$ has dimension at most $\dim A_*$. As a result, $\mathfrak{B}_s$ has dimension $\dim A_* + 1$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, we have a short exact sequence

$$0 \longrightarrow \overline{P_s} + T_1\mathfrak{B}'_s \longrightarrow \mathfrak{B}'_s \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s \longrightarrow 0 \ .$$

Then, setting $M := \overline{P_s} + T_1\mathfrak{B}'_s$ and denoting by M_* the localization of M at a maximal ideal of $\mathfrak{B}'_s$, containing T_1,

$$\mathrm{Ext}^j(\Bbbk, M_*) = 0$$

for $j \leqslant \dim A_*$ since $\mathfrak{B}'_s$ and $\mathfrak{B}_s/T_1\mathfrak{B}_s$ have dimension $\dim A_* + 1$ and $\dim A_*$. By Lemma 4.4(i), M is the direct sum $\overline{P_s}$ and $T_1\mathfrak{B}'_s$. So, denoting by $(T_1\mathfrak{B}'_s)_*$ the localization of $T_1\mathfrak{B}'_s$ at a maximal ideal of $\mathfrak{B}'_s$,

$$\mathrm{Ext}^j(\Bbbk, (T_1\mathfrak{B}'_s)_*) = 0$$

for $j \leqslant \dim A_*$ since $(T_1\mathfrak{B}'_s)_*$ is the localization of $\mathfrak{B}'_s$ at this maximal ideal when it does not contain T_1. As a result, by Lemma 4.4(ii), $\mathfrak{B}_s$ is Cohen-Macaulay since it has dimension $\dim A_* + 1$.

(ii) Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{B}_s$, containing $a_1, \ldots, a_m$. Since A_* is embedded in $\mathfrak{B}_s$, $\mathfrak{q} \cap A_*$ is a prime ideal of A_* containing $a_1, \ldots, a_m$. As A_* is Cohen-Macaulay and $a_1, \ldots, a_m$ is a regular sequence in A_*, $\mathfrak{q} \cap A_*$ has height at least m and $A_*/\mathfrak{q} \cap A_*$ has dimension at most $\dim A_* - m$ by [MA86, Ch. 6, Theorem 17.4]. Then $\mathfrak{B}_s/\mathfrak{q}$ has dimension at most $\dim A_* + 1 - m$ since the fraction field of $\mathfrak{B}_s/\mathfrak{q}$ is generated by the fraction field of $A_*/\mathfrak{q} \cap A_*$ and the image of T_1 by the quotient morphism $B_s \longrightarrow \mathfrak{B}_s/\mathfrak{q}$. As a result, by (i) and [MA86, Ch. 6, Theorem 17.4], $\mathfrak{q}$ has height at least m. As a minimal prime ideal of $\mathfrak{B}_s$ containing m elements, $\mathfrak{q}$ has height at most m. Hence all minimal prime ideal of $\mathfrak{B}_s$, containing $a_1, \ldots, a_m$, has height m. So, by (i) and [MA86, Ch. 6, Theorem 17.4], $a_1, \ldots, a_m$ is a regular sequence in the localization of $\mathfrak{B}_s$ at $\mathfrak{p}$. □

4.3. **Normality and Cohen-Macaulayness property for X_R.** Let V_0 be a fixed point under the action of R in X_R and $\beta_1, \ldots, \beta_d$ the elements of $\mathcal{R}_{V_0}$. By Corollary 2.22(ii), $\beta_1, \ldots, \beta_d$ is a basis of $\mathfrak{t}^*$. Let $t_1, \ldots, t_d$ be the dual basis. Denote by m the codimension of V_0 in $\mathfrak{a}$. According to Lie's Theorem, for $m > 0$, the elements $\gamma_1, \ldots, \gamma_m$ of $\mathcal{R} \setminus \{\beta_1, \ldots, \beta_d\}$ can be ordered so that

$$\mathfrak{a}_i := V_0 \oplus \mathfrak{a}^{\gamma_1} \oplus \cdots \oplus \mathfrak{a}^{\gamma_i}$$

is an algebra of codimension $m - i$ of $\mathfrak{a}$ for $i = 1, \ldots, m$. Set:

$$\mathcal{R}' := \mathcal{R} \setminus \{\gamma_m\}, \qquad \mathfrak{a}' = \mathfrak{a}_{m-1}, \qquad \mathfrak{r}' := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}'}, \qquad \pi' := \pi_{\mathfrak{t},\mathfrak{a}'}, \qquad R' := R_{\mathfrak{t},\mathfrak{a}'}, \qquad A' := A_{\mathfrak{t},\mathfrak{a}'},$$

$$E := \bigoplus_{i=1}^{m} \mathfrak{a}^{\gamma_i}, \qquad E' := E \cap \mathfrak{a}'.$$

Denote by κ the map

$$\mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t}) \xrightarrow{\ \kappa\ } \mathrm{Gr}_d(\mathfrak{r}) \ , \qquad \varphi \longmapsto \mathrm{span}(\{v + \varphi(v) \mid v \in V_0\}).$$

Then κ is an isomorphism from $\mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t})$ onto an affine open neighbourhood of V_0 in $\mathrm{Gr}_d(\mathfrak{r})$. Moreover, there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Bbbk}(V_0, \Bbbk x_{\gamma_m}) \longrightarrow \mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t}) \xrightarrow{p} \mathrm{Hom}_{\Bbbk}(V_0, E' \oplus \mathfrak{t}) \longrightarrow 0 \ .$$

Let Ω and Ω' be the inverse images by κ of the intersections of the image of κ with X_R and $X_{R'}$ respectively. For φ in Ω and $i = 1, \ldots, d$

$$\varphi(v_i) = \sum_{i=1}^{d} z_{i,j}(\varphi) t_j + \sum_{j=1}^{m} a_{i,j}(\varphi) x_{\gamma_j}$$

so that the $z_{i,j}$'s, $1 \leqslant i, j \leqslant d$ and the $a_{i,j}$'s, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$ are regular functions on Ω.

Let ψ be the map

$$\Bbbk \times \Omega' \xrightarrow{\quad\psi\quad} X_R \ , \qquad (s, \varphi) \longmapsto \exp(s \,\mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi).$$

Lemma 4.7. *Let O be the subset of elements (s, φ) of $\Bbbk \times \Omega'$ such that $\psi(s, \varphi)$ is in $\kappa(\Omega)$.*

(i) *The subset O of $\Bbbk \times \Omega'$ is open and contains $\{0\} \times \Omega'$.*

(ii) *The map*

$$O \xrightarrow{\quad\overline{\psi}\quad} \Omega \ , \qquad (s, \varphi) \longmapsto \kappa^{-1} \circ \psi(s, \varphi)$$

is a birational morphism from O to Ω. In particular, the function $(s, \varphi) \mapsto s$ is in $\Bbbk(\Omega)$.

Proof. (i) As $\kappa(\Omega)$ is an open neighborhood of V_0 in X_R, O is an open subset of $\Bbbk \times \Omega'$, containing $\{0\} \times \Omega'$ since ψ is a regular map such that $\psi(0, \varphi) = \kappa(\varphi)$ for all φ in Ω'.

(ii) Let Ω^c be the subset of elements φ of Ω such that $\kappa(\varphi)$ is in $A.\mathfrak{t}$. Then Ω^c is a dense open subset of Ω. Let O' be the inverse image of Ω^c by $\overline{\psi}$. Let (s, φ) and (s', φ') be in O' such that $\overline{\psi}(s, \varphi) = \overline{\psi}(s', \varphi')$, that is

$$\exp(s \,\mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi) = \exp(s' \mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi') \quad \text{whence} \quad \exp((s - s')\mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi) = \kappa(\varphi').$$

According to the above notations, for $i = 1, \ldots, d$,

$$\varphi(v_i) = \sum_{j=1}^{d} z_{i,j}(\varphi) t_j + \sum_{j=1}^{m-1} a_{i,j}(\varphi) x_{\gamma_j}.$$

Since $\kappa(\varphi)$ is in $A.\mathfrak{t}$,

$$\det([z_{i,j}(\varphi), 1 \leqslant i, j \leqslant d]) \neq 0.$$

For $i = 1, \ldots, d$,

$$\exp((s - s')\mathrm{ad}\, x_{\gamma_m})(\sum_{j=1}^{d} z_{i,j}(\varphi) t_j) = \sum_{j=1}^{d} z_{i,j}(\varphi) t_j - (s - s')(\sum_{j=1}^{d} z_{i,j}(\varphi) \gamma_m(t_j)) x_{\gamma_m}.$$

For some j, $\gamma_m(t_j) \neq 0$, whence $s = s'$ since $\kappa(\varphi')$ is contained in $\mathfrak{r}'$. As a result, the restriction of $\overline{\psi}$ to O' is injective, whence the assertion since $\overline{\psi}$ is a dominant morphism. □

For $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, denote by $u_{i,\gamma}$ the function on Ω,

$$u_{i,j} := z_{i,1}\gamma(t_1) + \cdots + z_{i,d}\gamma(t_d).$$

Let $\mathfrak{A}$ be the subalgebra of $\Bbbk[\Omega]$ generated by the functions $z_{i,j}$'s, $1 \leqslant i, j \leqslant d$ and $a_{i,j}$'s, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m - 1$.

Lemma 4.8. *Let ι be the restriction morphism from Ω to Ω'.*

(i) *The restriction of ι to $\mathfrak{A}$ is an isomorphism onto $\Bbbk[\Omega']$.*

(ii) *For $1 \leqslant i, j \leqslant d$, $u_{i,\gamma_m}a_{j,m} - u_{j,\gamma_m}a_{i,m}$ is equal to 0.*

(iii) *For $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, if $\gamma(t_i) \neq 0$ then $u_{i,\gamma}$ is different from 0.*

Proof. (i) For $1 \leqslant i, j \leqslant d$, denote by $z'_{i,j}$ the restriction of $z_{i,j}$ to Ω' and for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m-1$ denote by $a'_{i,j}$ the restriction of $a_{i,j}$ to Ω'. Since $\Bbbk[\Omega']$ is generated by the functions

$$z'_{i,j},\ 1 \leqslant i, j \leqslant d \quad \text{and} \quad a'_{i,j},\ 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant m-1,$$

the restriction of ι to $\mathfrak{A}$ is surjective. Let $\mathfrak{p}$ be the kernel of the restriction of ι to $\mathfrak{A}$. It remains to prove $\mathfrak{p} = \{0\}$.

For $1 \leqslant i, j \leqslant d$ and $k = 1, \ldots, m-1$, denote by $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ the functions on $\Bbbk \times \Omega'$ such that

$$\exp(s \operatorname{ad} x_{\gamma_m})(v_i + \sum_{j=1}^{d} z'_{i,j}(\varphi)t_j + \sum_{k=1}^{m-1} a'_{i,k}(\varphi)x_{\gamma_k}) -$$
$$(\sum_{j=1}^{d} \overline{z}_{i,j}(s,\varphi)t_j - \sum_{j=1}^{d} s z_{i,j}(\varphi)\gamma_m(t_j)x_{\gamma_m} + \sum_{k=1}^{m-1} \overline{a}_{i,k}(s,\varphi)x_{\gamma_k}) \in V_0.$$

Then $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ are regular functions on $\Bbbk \times \Omega'$ as restrictions to $\Bbbk \times \Omega'$ of regular functions on $\Bbbk \times \operatorname{Hom}(V_0, E' \oplus \mathfrak{t})$. Let $\overline{\mathfrak{A}}$ be the subalgebra of $\Bbbk[\Omega'][s]$ generated by the functions

$$\overline{z}_{i,j}, i, j = 1, \ldots, d \quad \text{and} \quad \overline{a}_{i,k}, i = 1, \ldots, d, k = 1, \ldots, m-1.$$

Since $z'_{i,j}(\varphi) = \overline{z}_{i,j}(0,\varphi)$ and $a'_{i,k}(\varphi) = \overline{a}_{i,k}(0,\varphi)$ for all φ in Ω', the restriction to $\overline{\mathfrak{A}}$ of the quotient morphism $\Bbbk[\Omega'][s] \longrightarrow \Bbbk[\Omega']$ is surjective. As a result, $\overline{\mathfrak{A}}$ has dimension n or $n-1$ since Ω' and $\Bbbk[\Omega'][s]$ have dimension $n-1$ and n respectively. As $\exp(s \operatorname{ad} x_{\gamma_m})(v_i)$ is not necessarily equal to v_i,

$$p \circ \psi \neq (\overline{z}_{i,j}, \overline{a}_{i,j},\ 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant m-1).$$

Moreover, Ω' is contained in $p(\Omega)$ by Lemma 4.7(i) but the inclusion may be strict.

Claim 4.9. The algebra $\overline{\mathfrak{A}}$ has dimension $n-1$.

Proof. [Proof of Claim 4.9] There are two cases to consider:

(1) for $i = 1, \ldots, m-1$, $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_i}]$ is contained in V_0,

(2) for some i in $\{1, \ldots, m-1\}$, $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_i}]$ is not contained in V_0.

In the first case, $\overline{\mathfrak{A}} = \Bbbk[\Omega']$. Otherwise, denote by j the biggest integer such that $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_j}]$ is not contained in V_0 and $a'_{i,j} \neq 0$ for some $i = 1, \ldots, d$. Then, for some j' smaller than j, $\gamma_m + \gamma_j = \gamma_{j'}$. Furthermore, for $k < j$ such that $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_k}]$ is not contained in V_0, $\gamma_m + \gamma_k$ is in $\mathcal{R} \setminus \{\gamma_{j'}, \ldots, \gamma_m\}$. Then for $k \geqslant j'$ and $i = 1, \ldots, d$, $a'_{i,k} = \overline{a}_{i,k}$ and for all (s,φ) in $\Bbbk \times \Omega'$,

$$\overline{a}_{i,j'}(s,\varphi) = a'_{i,j'}(\varphi) + s a'_{i,j}(\varphi).$$

As a result, by induction on $m-k$, for $i = 1, \ldots, d$,

$$a'_{i,k} - \overline{a}_{i,k} \in s\overline{\mathfrak{A}}[s].$$

Hence $\Bbbk[\Omega'][s] = \overline{\mathfrak{A}}[s]$ and there exists a surjective morphism $\Bbbk[\Omega'] \longrightarrow \overline{\mathfrak{A}}$ so that $\overline{\mathfrak{A}}$ has dimension $n-1$. □

According to Lemma 4.7(ii), the comorphism of $\overline{\psi}$ is an embedding of $\Bbbk[\Omega]$ into $\Bbbk[O]$ and from this embedding results an isomorphism from $\Bbbk(\Omega)$ onto $\Bbbk(\Omega')(s)$. Moreover, $\overline{\mathfrak{A}}$ is the image of $\mathfrak{A}$ by this embedding so that $\mathfrak{A}$ has dimension $n-1$. As a result, $\mathfrak{p} = \{0\}$ since ι is surjective and Ω' has dimension $n-1$.

(ii) Let φ be in Ω. Since $\kappa(\varphi)$ is a commutative algebra, for $1 \leqslant i, j \leqslant d$,

$$0 = [v_i + \varphi(v_i), v_j + \varphi(v_j)] = [v_i, \varphi(v_j)] + [\varphi(v_i), v_j] + [\varphi(v_i), \varphi(v_j)].$$

The component on x_{γ_m} of the right hand side is

$$\sum_{k=1}^{d} (z_{i,k} a_{j,m}(\varphi) - z_{j,k} a_{i,m}(\varphi))[t_k, x_{\gamma_m}] = (u_{i,\gamma_m} a_{j,m} - u_{j,\gamma_m} a_{i,m})(\varphi) x_{\gamma_m},$$

whence the assertion.

(iii) Denote by R_0 the adjoint group of $\mathfrak{r}_0 := \mathfrak{t} + V_0$ and X_{R_0} the closure in $\mathrm{Gr}_d(\mathfrak{r}_0)$ of $R_0.\mathfrak{t}$. Let Ω_0 be the inverse image of X_{R_0} by κ. According to Lemma 4.1, for $i, j = 1, \ldots, d$, the restriction to Ω_0 of $z_{i,j}$ is equal to 0 if $j \neq i$, otherwise it is different from 0. As a result, for $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, the restriction of $u_{i,\gamma}$ to Ω_0 is equal to $\overline{z_{i,i}}\gamma(t_i)$ with $\overline{z_{i,i}}$ the restriction of $z_{i,i}$ to Ω_0, whence the assertion. □

For γ in $\mathfrak{t}^*$, set:

$$I_\gamma := \{j \in \{1, \ldots, d\} \mid \gamma(t_j) \neq 0\}.$$

Proposition 4.10. *Denote by $\Bbbk[\Omega]_0$ the localization of $\Bbbk[\Omega]$ at 0.*

(i) *The local algebra $\Bbbk[\Omega]_0$ is Cohen-Macaulay.*

(ii) *For γ in $\mathfrak{t}^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\Bbbk[\Omega]_0$ of elements of its maximal ideal.*

Proof. Prove the proposition by induction on m. By Lemma 4.1, for $m = 0$, $\Bbbk[\Omega]$ is a polynomial algebra of dimension d, generated by $z_{1,1,0}, \ldots, z_{d,d,0}$. Moreover, for $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, $u_{i,\gamma} = z_{i,i}\gamma(t_i)$, whence the proposition for $m = 0$. Suppose $m > 0$ and the proposition true for $m-1$ and use the notations of Lemma 4.8.

According to Lemma 4.8(i) and the induction hypothesis, the localization $\mathfrak{A}_*$ of $\mathfrak{A}$ at 0 is Cohen-Macaulay and for γ in $\mathfrak{t}^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\mathfrak{A}_*$ of elements of its maximal ideal. Denote by $\mathfrak{B}$ the polynomial algebra $\mathfrak{A}_*[T_i, i \in I_{\gamma_m}]$ and by P the ideal of $\mathfrak{B}$ generated by the sequence $u_{i,\gamma_m}T_j - u_{j,\gamma_m}T_i, (i,j) \in I_{\gamma_m}^2$. According to Condition (3) of Section 2, $s := |I_{\gamma_m}| \geqslant 2$. By Lemma 4.8(ii), $\Bbbk[\Omega]_0$ is a quotient of the localization at 0 of $\mathfrak{B}/P$ and by Lemma 4.2, P is a prime ideal of $\mathfrak{B}$. By Proposition 4.6(i), $\mathfrak{B}/P$ is an integral domain and a Cohen-Macaulay algebra of dimension n since $\Bbbk[\Omega']$ has dimension $n-1$. Hence $\Bbbk[\Omega]_0$ is the localization of $\mathfrak{B}/P$ at 0 since $\Bbbk[\Omega]_0$ is an integral domain of dimension n. As a result, $\Bbbk[\Omega]_0$ is Cohen-Macaulay and by Proposition 4.6(ii), for γ in $\mathfrak{t}^*$, the sequence $u_{i,\gamma}, i \in I_\gamma$ is regular in $\Bbbk[\Omega]_0$. □

Theorem 4.11. *The variety X_R is normal and Cohen-Macaulay.*

Proof. By Corollary 3.8, X_R is smooth in codimension 1. So, by Serre's normality criterion [Bou98, §1,no 10, Théorème 4], it suffices to prove that X_R is Cohen-Macaulay. According to [MA86, Ch. 8, Theorem 24.5], the set of points x of X_R such that $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay, is open. For x in X_R, the closure in X_R of $R.x$ contains a fixed point. So it suffices to prove that for x a fixed point under the action of R in X_R, $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay. Let V_0 and Ω be as in Lemma 4.7. Then Ω is an affine open neighborhood of V_0 in X_R. By Proposition 4.10(i), $\mathcal{O}_{\Omega,0}$ is Cohen-Macaulay, whence the theorem since κ is an isomorphism from Ω onto an open neighborhood of V_0 in X_R and $\kappa(0) = V_0$. □

4.4. **Nipotent cone and regular sequence in $\mathcal{O}_{\mathcal{E}}$.** Let $\beta_1,\ldots,\beta_d$ be a basis of $\mathfrak{t}^*$. For $i=1,\ldots,d$, denote again by β_i the element of $\mathfrak{r}^*$ extending β_i and equal to 0 on $\mathfrak{a}$. For Λ a complete subset of $\mathcal{R}$, denote by $\mathfrak{t}^{\#}_{\Lambda}$ a complement to $\mathfrak{t}_{\Lambda}$ in $\mathfrak{t}$ and set

$$R'_{\Lambda} := R_{\mathfrak{t}^{\#}_{\Lambda},\mathfrak{a}_{\Lambda}} \quad \text{and} \quad \mathcal{E}_{\Lambda} := \mathcal{E}_{\mathfrak{t}^{\#}_{\Lambda},\mathfrak{a}_{\Lambda}}.$$

For Y closed subset of $X_{R'_{\Lambda}}$, denote by $\mathcal{E}_{\Lambda,Y}$ the restriction of $\mathcal{E}_{\Lambda}$ to Y. Let $\mathcal{N}'_{\Lambda}$ be the image of the map

$$\mathcal{E}_{\Lambda,X_{R'_{\Lambda},\mathrm{n}}} \longrightarrow \mathcal{E}\ , \qquad (V,x)\longmapsto (V\oplus\mathfrak{t}_{\Lambda},x)$$

and $\mathcal{N}_{\Lambda}$ the closure in $\mathcal{E}$ of $R.\mathcal{N}'_{\Lambda}$.

Lemma 4.12. *For $i=1,\ldots,d$, let $\tilde{\beta}_i$ be the function on $\mathcal{E}$ defined by $\tilde{\beta}_i(V,x)=\beta_i(x)$. Denote by $\mathcal{N}$ the nullvariety of $\tilde{\beta}_1,\ldots,\tilde{\beta}_d$ in $\mathcal{E}$.*

(i) *For all complete subset Λ of $\mathcal{R}$, $\mathcal{N}_{\Lambda}$ is a subvariety of $\mathcal{N}$ of dimension at most n.*

(ii) *The varieyt $\mathcal{N}$ is the union of $\mathcal{N}_{\Lambda}, \Lambda\in\mathcal{P}_c(\mathcal{R})$.*

(iii) *The variety $\mathcal{N}$ is equidimensional of dimension n.*

Proof. (i) Since $\mathfrak{a}$ is the nullvariety of $\beta_1,\ldots,\beta_d$ in $\mathfrak{r}$, $\mathcal{N}$ is the intersection of $\mathcal{E}$ and $X_R\times\mathfrak{a}$. By definition $\mathcal{N}'_{\Lambda}$ is contained in $X_R\times\mathfrak{a}$. Hence $\mathcal{N}_{\Lambda}$ is contained in $\mathcal{N}$. By Proposition 3.7,

$$\dim\mathcal{N}'_{\Lambda} = \dim\mathfrak{t}^{\#}_{\Lambda} + \dim X_{R'_{\Lambda},\mathrm{n}} \leqslant \dim\mathfrak{a}_{\Lambda}.$$

Since the image of $X_{R'_{\Lambda},\mathrm{n}}$ by the map $V\mapsto V\oplus\mathfrak{t}_{\Lambda}$ is invariant by R_{Λ},

$$\dim\mathcal{N}_{\Lambda} \leqslant \dim\mathcal{N}'_{\Lambda} + \dim\mathfrak{a} - \dim\mathfrak{a}_{\Lambda} \leqslant \dim\mathfrak{a}.$$

(ii) Let ϖ_1 be the bundle projection of the vector bundle $\mathcal{E}$ over X_R and τ_1 the restriction to $\mathcal{E}$ of the projection $X_R\times\mathfrak{r}\longrightarrow\mathfrak{r}$. Let T be an irreducible component of $\mathcal{N}$. For all V in $\varpi_1(T)$, $\tau_1(\varpi_1^{-1}(V)\cap T)$ is a closed cone of $\mathfrak{a}$. Hence $\varpi_1(T)\times\{0\}$ is the intersection of T and $X_R\times\{0\}$ so that $\varpi_1(T)$ is a closed subset of X_R. Since $\mathcal{N}$ is the intersection of $\mathcal{E}$ and $X_R\times\mathfrak{a}$, $\mathcal{N}$ and its irreducible components are invariant under R. As a result, $\varpi_1(T)$ is invariant under R and by Proposition 3.1, for some complete subset Λ of $\mathcal{R}$ and for some closed subset of Z_{Λ} of $X_{R_{\Lambda}}$, $\varpi_1(T)=\overline{R.Z_{\Lambda}}$. Moreover, by Lemma 3.2, for some closed subset Z'_{Λ} of $X_{R'_{\Lambda},\mathrm{n}}$, Z_{Λ} is the image of Z'_{Λ} by the map $V\mapsto V\oplus\mathfrak{t}_{\Lambda}$. As a result,

$$\mathcal{E}_{\Lambda,Z'_{\Lambda}}\subset\mathcal{E}_{\Lambda,X_{R'_{\Lambda},\mathrm{n}}} \quad \text{and} \quad \varpi_1^{-1}(Z_{\Lambda})\cap X_R\times\mathfrak{a}\subset\mathcal{N}'_{\Lambda}.$$

Then T is contained in $\mathcal{N}_{\Lambda}$, whence the assertion by (i).

(iii) By (i) and (ii), since $\mathcal{R}$ is finite, the irreducible components of $\mathcal{N}$ have dimension at most n. As the nullvariety of d functions on the irreducible variety $\mathcal{E}_{X_R}$, the irreducible components of $\mathcal{N}$ have dimension at least n, whence the assertion. □

For x in $\mathcal{E}$, denote by I_x the subset of elements i of $\{1,\ldots,d\}$ such that $\tilde{\beta}_i(x)=0$.

Corollary 4.13. *For all x in $\mathcal{E}$, the sequence $\tilde{\beta}_i, i\in I_x$ is regular in $\mathcal{O}_{\mathcal{E},x}$.*

Proof. According to Lemma 4.12, for all subset I of $\{1,\ldots,d\}$, the nullvariety of $\tilde{\beta}_i, i\in I$ in $\mathcal{E}$ is equidimensional of dimension $n+d-|I|$. By Theorem 4.11 and Lemma B.1(iii), $\mathcal{E}$ is Cohen-Macaulay as a vector bundle over a Cohen-Macaulay variety, whence the corollary by [MA86, Ch. 6, Theorem 17.4]. □

5. Rational singularities for solvable Lie algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d. Denote by $\mathcal{C}_{\mathfrak{t},*}$ the full subcategory of $\mathcal{C}_{\mathfrak{t}}$ whose objects $\mathfrak{a}$ satisfy the following condition:

(4) there exist regular maps $\varepsilon_1, \ldots, \varepsilon_d$ from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ such that $\varepsilon_1(x), \ldots, \varepsilon_d(x)$ is a basis of $\mathfrak{r}^x_{\mathfrak{t},\mathfrak{a}}$ for all x in $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}_{\rm reg}}$.

According to [Ko63, Theorem 9], $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{b},*}$.

Lemma 5.1. *Let $\mathfrak{a}$ be in $\mathcal{C}_{\mathfrak{t},*}$ and $\mathfrak{a}'$ an ideal of $\mathfrak{t}+\mathfrak{a}$, contained in $\mathfrak{a}$ and containing a fixed point under the action of $R_{\mathfrak{t},\mathfrak{a}}$ in $X_{R_{\mathfrak{t},\mathfrak{a}}}$. Then $\mathfrak{a}'$ is in $\mathcal{C}_{\mathfrak{t},*}$.*

Proof. Set $\mathfrak{r} := \mathfrak{t}+\mathfrak{a}$ and $\mathfrak{r}' := \mathfrak{t}+\mathfrak{a}'$. According to Corollary 2.22(ii), $\mathfrak{a}'$ is in $\mathcal{C}_{\mathfrak{t}}$ since it is in $\mathcal{C}'_{\mathfrak{t}}$. Set $\mathfrak{t}_{\rm reg} := \mathfrak{r}_{\rm reg} \cap \mathfrak{t}$. As $\mathcal{R}_{\mathfrak{t},\mathfrak{a}'}$ is contained in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\mathfrak{t}_{\rm reg}$ is contained in $\mathfrak{r}'_{\rm reg}$ by Lemma 3.3(i). Then $\mathfrak{r}'_{\rm reg}$ is contained in $\mathfrak{r}_{\rm reg}$ and for all x in $A_{\mathfrak{t},\mathfrak{a}'}.\mathfrak{t}_{\rm reg}$, $\mathfrak{r}^x = \mathfrak{r}'^x$ since $A_{\mathfrak{t},\mathfrak{a}'}.\mathfrak{t}_{\rm reg}$ is a dense open subset of $\mathfrak{r}'$ by Lemma 3.3(i). So, for all regular map ε from $\mathfrak{r}$ to $\mathfrak{r}$ such that $[x,\varepsilon(x)] = 0$ for all x in $\mathfrak{r}$, $\varepsilon(x)$ is in $\mathfrak{r}'$ for all x in $\mathfrak{r}'$, whence the lemma. □

Let $\mathfrak{a}$ be in $\mathcal{C}_{\mathfrak{t},*}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is Gorenstein with rational singularities.

For k positive integer, set:

$$\mathcal{E}^{(k)} := \{(u, x_1, \ldots, x_k) \in X_R \times \mathfrak{r}^k \mid u \ni x_1, \ldots, u \ni x_k\}$$

and denote by $\mathfrak{X}_{R,k}$ the image of $\mathcal{E}^{(k)}$ by the projection

$$(u, x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k).$$

Since X_R is a projective variety, $\mathfrak{X}_{R,k}$ is a closed subset of $\mathfrak{r}^k$, invariant under the diagonal action of R in $\mathfrak{r}^k$.

5.1. Differential forms on some smooth open subsets of $\mathfrak{X}_{R,k}$.

For $j = 1, \ldots, k$, let $V^{(k)}_j$ be the subset of elements of $\mathfrak{X}_{R,k}$ whose j-th component is in $\mathfrak{r}_{\rm reg}$.

Lemma 5.2. *For $j = 1, \ldots, k$, $V^{(k)}_j$ is a smooth open subset of $\mathfrak{X}_{R,k}$. Moreover, $\Omega_{V^{(k)}_j}$ has a global section without zero.*

Proof. Denoting by σ_j the automorphism of $\mathfrak{r}_k$ which permutes the first and the j-th component, $\mathfrak{X}_{R,k}$ is invariant under σ_j and $\sigma_j(V^{(k)}_1) = V^{(k)}_j$ so that we can suppose $j = 1$. Moreover, for $k = 1$, $\mathfrak{X}_{R,k} = \mathfrak{r}$ so that we can suppose $k \geqslant 2$. By definition, $V^{(k)}_1$ is the intersection of $\mathfrak{r}_{\rm reg} \times \mathfrak{r}^{k-1}$ and $\mathfrak{X}_{R,k}$. Hence $V^{(k)}_1$ is an open susbet of $\mathfrak{X}_{R,k}$ since $\mathfrak{r}_{\rm reg}$ is an open subset of $\mathfrak{r}$.

Let $\varepsilon_1, \ldots, \varepsilon_d$ satisfying Condition (4) with respect to $\mathfrak{r}$. Let θ be the map

$$\mathfrak{r}_{\rm reg} \times \mathrm{M}_{k-1,d}(\Bbbk) \xrightarrow{\ \theta\ } \mathfrak{r}^k, \qquad (x, a_{i,j}, 2 \leqslant i \leqslant k, 1 \leqslant j \leqslant d) \longmapsto (x, \sum_{j=1}^{d} a_{i,j}\varepsilon_j(x)).$$

Since for all $(x, x_2, \ldots, x_k)$ in $V^{(k)}_1$, $x_2, \ldots, x_k$ are in $\mathfrak{r}^x$, θ is a bijective map onto $V^{(k)}_1$. The open subset $\mathfrak{r}_{\rm reg}$ has a cover by open subsets V such that for some $e_1, \ldots, e_n$ in $\mathfrak{r}$, $\varepsilon_1(x), \ldots, \varepsilon_d(x), e_1, \ldots, e_n$ is a basis of $\mathfrak{r}$ for all x in V. Then there exist regular functions $\varphi_1, \ldots, \varphi_d$ on $V \times \mathfrak{r}$ such that

$$v - \sum_{j=1}^{\ell} \varphi_j(x, v)\varepsilon_j(x) \in \mathrm{span}(\{e_1, \ldots, e_n\})$$

for all (x,v) in $V\times\mathfrak{r}$, so that the restriction of θ to $V\times \mathrm{M}_{k-1,d}(\Bbbk)$ is an isomorphism onto $\mathfrak{X}_{R,k}\cap V\times\mathfrak{r}^{k-1}$ whose inverse is

$$(x_1,\dots,x_k)\longmapsto (x_1,((\varphi_1(x_1,x_i),\dots,\varphi_d(x_1,x_i)),i=2,\dots,k))$$

As a result, θ is an isomorphism and $V_1^{(k)}$ is a smooth variety. Since $\mathfrak{r}_{\rm reg}$ is a smooth open subset of the vector space $\mathfrak{r}$, there exists a regular differential form ω of top degree on $\mathfrak{r}_{\rm reg}\times\mathrm{M}_{k-1,\ell}(\Bbbk)$, without zero. Then $\theta_*(\omega)$ is a regular differential form of top degree on $V_1^{(k)}$, without zero. □

For $k\geqslant 2$ set:

$$V^{(k)}:=V_1^{(k)}\cup V_2^{(k)}\quad\text{and}\quad V_{1,2}^{(k)}:=V_1^{(k)}\cap V_2^{(k)}.$$

For $2\leqslant k'\leqslant k$, the projection

$$\mathfrak{r}^k\longrightarrow\mathfrak{r}^{k'}\ ,\qquad (x_1,\dots,x_k)\longmapsto(x_1,\dots,x_{k'})$$

induces the projection

$$\mathfrak{X}_{R,k}\longrightarrow\mathfrak{X}_{R,k'}\ ,\qquad V_j^{(k)}\longrightarrow V_j^{(k')}$$

for $j=1,\dots,k'$.

Lemma 5.3. *Suppose $k\geqslant 2$. Let ω be a regular differential form of top degree on $V_1^{(k)}$, without zero. Denote by ω' its restriction to $V_{1,2}^{(k)}$.*

(i) *For φ in $\Bbbk[V_1^{(k)}]$, if φ has no zero then φ is in $\Bbbk^*$.*

(ii) *For some invertible element ψ of $\Bbbk[V_{1,2}^{(2)}]$, $\omega'=\psi\sigma_{2*}(\omega')$.*

(iii) *The function $\psi(\psi\circ\sigma_2)$ on $V_{1,2}^{(k)}$ is equal to* 1.

Proof. The existence of ω results from Lemma 5.2.

(i) According to Lemma 5.2, there is an isomorphism θ from $\mathfrak{r}_{\rm reg}\times\mathrm{M}_{k-1,d}(\Bbbk)$ onto $V_1^{(k)}$. Since φ is invertible, $\varphi\circ\theta$ is an invertible element of $\Bbbk[\mathfrak{r}_{\rm reg}]$. According to Lemma 3.3(iii), $\Bbbk[\mathfrak{r}_{\rm reg}]=\Bbbk[\mathfrak{r}]$. Hence φ is in $\Bbbk^*$.

(ii) The open subset $V_{1,2}^{(k)}$ is invariant under σ_2 so that ω' and $\sigma_{2*}(\omega')$ are regular differential forms of top degree on $V_{1,2}^{(k)}$, without zero. Then for some invertible element ψ of $\Bbbk[V_{1,2}^{(k)}]$, $\omega'=\psi\sigma_{2*}(\omega')$. Let O_2 be the set of elements $(x,a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant d)$ of $\mathfrak{r}_{\rm reg}\times\mathrm{M}_{k-1,d}(\Bbbk)$ such that

$$a_{1,1}\varepsilon_1(x)+\cdots+a_{1,\ell}\varepsilon_\ell(x)\in\mathfrak{r}_{\rm reg}.$$

Then O_2 is the inverse image of $V_{1,2}^{(k)}$ by θ. As a result, $\Bbbk[V_{1,2}^{(k)}]$ is a polynomial algebra over $\Bbbk[V_{1,2}^{(2)}]$ since for $k=2$, O_2 is the inverse image by θ of $V_{1,2}^{(2)}$. Hence ψ is in $\Bbbk[V_{1,2}^{(2)}]$ since ψ is invertible.

(iii) Since the restriction of σ_2 to $V_{1,2}^{(k)}$ is an involution,

$$\sigma_{2*}(\omega')=(\psi\circ\sigma_2)\omega'=(\psi\circ\sigma_2)\psi\sigma_{2*}(\omega'),$$

whence $(\psi\circ\sigma_2)\psi=1$. □

Corollary 5.4. *The function ψ is invariant under the action of R in $V_{1,2}^{(k)}$ and for some sequence $m_\alpha,\alpha\in\mathcal{R}$ in $\mathbb{Z}$,*

$$\psi(x_1,\dots,x_k)=\pm\prod_{\alpha\in\mathcal{R}}(\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all $(x_1,\dots,x_k)$ in $\mathfrak{t}_{\rm reg}^2\times\mathfrak{t}^{k-2}$.

Proof. First of all, since $V_1^{(k)}$ and $V_2^{(k)}$ are invariant under the action of R in $\mathfrak{X}_{R,k}$, so is $V_{1,2}^{(k)}$. Let g be in R. As ω has no zero, $g.\omega = p_g\omega$ for some invertible element p_g of $\Bbbk[V_1^{(k)}]$. By Lemma 5.3(i), p_g is in $\Bbbk^*$. Since σ_2 is a R-equivariant isomorphism from $V_1^{(k)}$ onto $V_2^{(k)}$,

$$g.\sigma_{2*}(\omega) = p_g\sigma_{2*}(\omega) \quad \text{and} \quad p_g\omega' = g.\omega' = (g.\psi)g.\sigma_{2*}(\omega') = p_g(g.\psi)\sigma_{2*}(\omega'),$$

whence $g.\psi = \psi$.

The open subset $\mathfrak{t}^2_{\rm reg}$ of $\mathfrak{t}^2$ is the complement to the nullvariety of the function

$$(x,y) \longmapsto \prod_{\alpha\in\mathcal{R}} \alpha(x)\alpha(y).$$

Then, by Lemma 5.3(ii), for some a in $\Bbbk^*$ and for some sequences $m_\alpha, \alpha \in \mathcal{R}$ and $n_\alpha, \alpha \in \mathcal{R}$ in $\mathbb{Z}$,

$$\psi(x_1,\dots,x_k) = a\prod_{\alpha\in\mathcal{R}} \alpha(x_1)^{m_\alpha}\alpha(x_2)^{n_\alpha},$$

for all $(x_1,\dots,x_k)$ in $\mathfrak{t}^2_{\rm reg}\times\mathfrak{t}^{k-2}$. By Lemma 5.3(iii),

$$a^2\prod_{\alpha\in\mathcal{R}} \alpha(x)^{m_\alpha+n_\alpha}\alpha(y)^{m_\alpha+n_\alpha} = 1,$$

for all (x,y) in $\mathfrak{t}^2_{\rm reg}$. Hence $a^2 = 1$ and $m_\alpha + n_\alpha = 0$ for all α in $\mathcal{R}$. □

According to Lemma 3.5(i), for α in $\mathcal{R}$, θ_α is a bijective regular map from $\mathbb{P}^1(\Bbbk)$ onto the closed subset Z_α of X_R such that $\theta_\alpha(\infty) = V_\alpha$. Recall that x_α is a generator of $\mathfrak{a}^\alpha$ and h_α is an element of $\mathfrak{t}$ such that $\alpha(h_\alpha) = 1$. Denote by $\mathfrak{t}'_\alpha$ the subset of elements x of $\mathfrak{t}_\alpha$ such that $\gamma(x) \neq 0$ for all γ in $\mathcal{R}\setminus\{\alpha\}$. According to Condition (3) of Section 2, $\mathfrak{t}'_\alpha$ is a dense open subset of $\mathfrak{t}_\alpha$. Let $x_{-\alpha}$ be in $\mathfrak{r}^*$ orthogonal to $\mathfrak{t}+\mathfrak{a}^\gamma$ for all γ in $\mathcal{R}\setminus\{\alpha\}$ and such that $x_{-\alpha}(x_\alpha) = 1$.

Lemma 5.5. *Suppose $k \geqslant 2$. Let α be in $\mathcal{R}$, x_0 and y_0 in $\mathfrak{t}'_\alpha$. Set:*

$E := \Bbbk x_0 \oplus \Bbbk h_\alpha \oplus \mathfrak{a}^\alpha, \quad E_* := x_0 \oplus \Bbbk h_\alpha \oplus \mathfrak{a}^\alpha, \quad E_{*,1} := x_0 \oplus \Bbbk h_\alpha \oplus (\mathfrak{a}^\alpha\setminus\{0\}), \quad E_{*,2} = y_0 \oplus \Bbbk h_\alpha \oplus (\mathfrak{a}^\alpha\setminus\{0\}).$

(i) *For x in E_*, $\mathfrak{r}^x$ is contained in $\mathfrak{t}_\alpha + E$.*

(ii) *For V subspace of dimension d of $\mathfrak{t}_\alpha + E$, V is in X_R if and only if it is in Z_α.*

(iii) *The intersection of $E_{*,1}\times E_{*,2}$ and $\mathfrak{X}_{R,2}$ is the nullvariety of the function*

$$(x,y) \longmapsto x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y)$$

on $E_{*,1}\times E_{*,2}$.

Proof. (i) If x is regular semisimple, its component on h_α is different from 0 so that $\mathfrak{r}^x = \theta_\alpha(z)$ for some z in $\Bbbk$. Suppose that x is not regular semisimple. Then x is in $x_0 + \mathfrak{a}^\alpha$. Hence $\mathfrak{r}^x$ is contained in $\mathfrak{t}_\alpha + E$ since so is $\mathfrak{r}^{x_0}$.

(ii) All element of Z_α is contained in $\mathfrak{t}_\alpha + E$. Let V be an element of X_R, contained in $\mathfrak{t}_\alpha + E$. According to Corollary 2.22(i), V is an algebraic commutative subalgebra of dimension d of $\mathfrak{r}$. By (i), $V = \theta_\alpha(z)$ for some z in $\Bbbk$ if V is in $A.\mathfrak{t}$. Otherwise, x_α is in V. Then $V = \theta_\alpha(\infty)$ since $\theta_\alpha(\infty)$ is the centralizer of x_α in $\mathfrak{t}_\alpha + E$.

(iii) Let (x,y) be in $E_{*,1}\times E_{*,2}\cap\mathfrak{X}_{R,2}$. By definition, for some V in X_R, x and y are in V. By (i) and (ii), $V = \theta_\alpha(z)$ for some z in $\mathbb{P}^1(\Bbbk)$. For z in $\Bbbk$,

$$x = x_0 + s(h_\alpha - zx_\alpha) \quad \text{and} \quad y = y_0 + s'(h_\alpha - zx_\alpha)$$

for some s, s' in $\Bbbk$, whence the equality of the assertion. For $z = \infty$,

$$x = x_0 + sx_\alpha \quad \text{and} \quad y = y_0 + s'x_\alpha$$

for some s, s' in $\Bbbk$ so that $\alpha(x) = \alpha(y) = 0$. Conversely, let (x, y) be in $E_{*,1} \times E_{*,2}$ such that

$$x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y) = 0.$$

If $\alpha(x) = 0$ then $\alpha(y) = 0$ and x and y are in $V_\alpha = \theta_\alpha(\infty)$. If $\alpha(x) \neq 0$, then $\alpha(y) \neq 0$ and

$$x \in \theta_\alpha(-\frac{x_{-\alpha}(x)}{\alpha(x)}) \quad \text{and} \quad y \in \theta_\alpha(-\frac{x_{-\alpha}(x)}{\alpha(x)}),$$

whence the assertion. □

Set $V^{(1)} := \mathfrak{r}_{\rm reg}$.

Proposition 5.6. *For k positive integer, there exists on $V^{(k)}$ a regular differential form of top degree without zero.*

Proof. For $k = 1$, it is true since $\mathfrak{r}_{\rm reg}$ is an open subset of the vector sapce $\mathfrak{r}$. So we can suppose $k \geqslant 2$. According to Corollary 5.4, it suffices to prove $m_\alpha = 0$ for all α in $\mathcal{R}$. Indeed, if so, by Corollary 5.4, $\psi = \pm 1$ on the open subset $R.(\mathfrak{t}_{\rm reg}^2 \times \mathfrak{t}^{k-2})$ of $V^{(k)}$ so that $\psi = \pm 1$ on $V_{1,2}^{(k)}$. Then, by Lemma 5.3(ii), ω and $\pm\sigma_{2*}(\omega)$ have the same restriction to $V_{1,2}^{(k)}$ so that there exists a regular differential form of top degree $\tilde{\omega}$ on $V^{(k)}$ whose restrictions to $V_1^{(k)}$ and $V_2^{(k)}$ are ω and $\pm\sigma_{2*}(\omega)$ respectively. Moreover, $\tilde{\omega}$ has no zero since so has ω.

Since ψ is in $\Bbbk[V_{1,2}^{(2)}]$ by Lemma 5.3(ii), we can suppose $k = 2$. Let α be in $\mathcal{R}$, E, E_*,$E_{*,1}$, $E_{*,2}$ as in Lemma 5.3. Suppose $m_\alpha \neq 0$. A contradiction is expected. The restriction of ψ to $E_{*,1} \times E_{*,2} \cap V_{1,2}^{(2)}$ is given by

$$\psi(x, y) = a x_{-\alpha}(x)^m x_{-\alpha}(y)^n,$$

with a in $\Bbbk^*$ and (m, n) in $\mathbb{Z}^2$ since ψ is an invertible element of $\Bbbk[V_{1,2}^{(2)}]$. According to Lemma 5.3(iii), $n = -m$ and $a = \pm 1$. Interchanging the role of x and y, we can suppose m in $\mathbb{N}$. For (x, y) in $E_{*,1} \times E_{*,2} \cap V_{1,2}^{(2)}$ such that $\alpha(x) \neq 0$, $\alpha(y) \neq 0$ and

$$\psi(x, y) = \pm x_{-\alpha}(x)^m (\frac{x_{-\alpha}(x)\alpha(y)}{\alpha(x)})^{-m} = \pm\alpha(x)^m \alpha(y)^{-m}.$$

As a result, by Corollary 5.4, for x in $x_0 + \Bbbk^* h_\alpha$ and y in $y_0 + \Bbbk^* h_\alpha$,

(1) $$\pm\alpha(x)^m \alpha(y)^{-m} = \pm \prod_{\gamma\in\mathcal{R}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma}.$$

For γ in $\mathcal{R}$,

$$\gamma(x) = \gamma(x_0) + \alpha(x)\gamma(h_\alpha) \quad \text{and} \quad \gamma(y) = \gamma(y_0) + \alpha(y)\gamma(h_\alpha).$$

Since m is in $\mathbb{N}$,

(2) $$\pm\alpha(x)^m \prod_{\substack{\gamma\in\mathcal{R} \\ m_\gamma>0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R} \\ m_\gamma<0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{-m_\gamma} =$$
$$\pm\alpha(y)^m \prod_{\substack{\gamma\in\mathcal{R} \\ m_\gamma>0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R} \\ m_\gamma<0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{-m_\gamma}.$$

For m_α positive, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm\alpha(x)^m \alpha(y)^{m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\} \\ m_\gamma>0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\} \\ m_\gamma<0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm\, \alpha(y)^m \alpha(x)^{m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\} \\ m_\gamma>0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\} \\ m_\gamma<0}} \gamma(y_0)^{-m_\gamma}$$

respectively and for m_α negative, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm\alpha(x)^{m+m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm\,\alpha(y)^{m+m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(y_0)^{-m_\gamma}$$

respectively. From the equality of these terms, we deduce $m = \pm m_\alpha$ and

$$\prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(x_0)^{-m_\gamma} = \pm \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(y_0)^{-m_\gamma}.$$

Since the last equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, this equality remains true for all (x_0, y_0) in $\mathfrak{t}_\alpha \times \mathfrak{t}_\alpha$. As a result, as the degrees in $\alpha(x)$ of the left and right sides of Equality (2) are the same,

$$m - \sum_{\substack{\gamma\in\mathcal{R}\\ m_\gamma<0 \text{ and } \gamma(h_\alpha)\neq 0}} m_\gamma = \sum_{\substack{\gamma\in\mathcal{R}\\ m_\gamma>0 \text{ and } \gamma(h_\alpha)\neq 0}} m_\gamma. \tag{3}$$

Suppose $m = m_\alpha$. By Equality (1),

$$\prod_{\gamma\in\mathcal{R}\setminus\{\alpha\}} \gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm 1.$$

Since this equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, it holds for all (x, y) in $\mathfrak{t}_{\rm reg} \times \mathfrak{t}_{\rm reg}$. Hence $m_\gamma = 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$ and $m = 0$ by Equality (3). It is impossible since $m_\alpha \neq 0$. Hence $m = -m_\alpha$. Then, by Equality (1)

$$\prod_{\gamma\in\mathcal{R}\setminus\{\alpha\}} \gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm\alpha(x)^{2m}\alpha(y)^{-2m}.$$

Since this equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, it holds for all (x, y) in $\mathfrak{t}_{\rm reg} \times \mathfrak{t}_{\rm reg}$. Then $m = 0$, whence the contradiction. □

5.2. **Rational singularities and Gorensteinness of X_R.** For Y subvariety of $\mathrm{Gr}_d(\mathfrak{r})$, denote by $\mathcal{E}_Y$ the restriction to Y of the tautological vector bundle of rank d over $\mathrm{Gr}_d(\mathfrak{r})$. In particular, for Y contained in X_R, $\mathcal{E}_Y$ is a subvariety of $\mathcal{E}$. For k positive integer, denote by τ_k and ϖ_k the restrictions to $\mathcal{E}^{(k)}$ of the canonical projections

$$X_R \times \mathfrak{r}^k \xrightarrow{\tau_k} \mathfrak{r}^k \quad \text{and} \quad X_R \times \mathfrak{r}^k \xrightarrow{\varpi_k} X_R\ .$$

Lemma 5.7. (i) *The morphism τ_k is a projective and birational morphism onto $\mathfrak{X}_{R,k}$.*

(ii) *The sets $V^{(k)}$ and $\tau_k^{-1}(V^{(k)})$ are smooth open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$. Moreover, for $k \geqslant 2$, they are big open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$.*

(iii) *The restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is an isomorphism onto $V^{(k)}$.*

Proof. Since X_R is a projective variety, τ_k is projective and its image is $\mathfrak{X}_{R,k}$ by definition. For $(x_1, \ldots, x_k)$ in $V^{(k)}$ and $(u, x_1, \ldots, x_k)$ in $\tau_k^{-1}((x_1, \ldots, x_k))$, $u = \mathfrak{r}^{x_1}$ if x_1 is in $\mathfrak{r}_{\rm reg}$ and $u = \mathfrak{r}^{x_2}$ if x_2 is in $\mathfrak{r}_{\rm reg}$. As a result, the restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is a bijective morphism onto $V^{(k)}$. Hence τ_k is a birational morphism and by Zariski's Main Theorem [Mu88, §9], this restriction is an isomorphism since $V^{(k)}$ is a smooth variety by Lemma 5.2. So it remains to prove that for $k \geqslant 2$, $\tau_k^{-1}(V^{(k)})$ is a big open subset of $\mathcal{E}^{(k)}$

Suppose that $\mathcal{E}^{(k)} \setminus \tau_k^{-1}(V^{(k)})$ has an irreducible component Σ of dimension $\dim \mathcal{E}^{(k)} - 1$. A contradiction is expected. Since $\mathcal{E}^{(k)}$ and $\tau_k^{-1}(V_k)$ are invariant under the automorphisms of $X_R \times \mathfrak{r}^k$,

$$(u, x_1, \ldots, x_k) \longmapsto (u, tx_1, \ldots, tx_k), \qquad (t \in \Bbbk^*),$$

so is Σ. Then $\Sigma \cap X_R \times \{0\} = \varpi_k(\Sigma) \times \{0\}$ so that $\varpi_k(\Sigma)$ is a closed subset of X_R. Since $\dim \Sigma = \dim \mathcal{E}^{(k)} - 1$, $\dim \varpi_k(\Sigma) \geqslant \dim X_R - 1$. Suppose $\dim \Sigma = \dim X_R - 1$. Then for all u in $\varpi_k(\Sigma)$, $\{u\} \times u^k$ is in Σ. It is impossible since for all u in a dense open subset of $\varpi_k(\Sigma)$, $u = \mathfrak{r}^x$ for some x in $\mathfrak{r}_{\rm reg}$ by Corollary 3.8. Hence $\varpi_k(\Sigma) = X_R$. Then for all u in a dense open subset of X'_R, $\{u\} \times u^k \cap \Sigma$ has codimension 1 in $\{u\} \times u^k$. Since the image of $\{u\} \times u^k \cap \Sigma$ by the projection

$$(u, x_1, \dots, x_k) \longmapsto x_1$$

is not dense in u, for all x_1 in a dense open subset of its image, $\{u\} \times \{x_1\} \times u^{k-1}$ is contained in Σ, whence the contradiction since $u \cap \mathfrak{r}_{\rm reg}$ is not empty. □

By definition, $\mathcal{E}^{(k)}$ is the inverse image of X_R by the bundle projection of the vector bundle

$$\{(u, x_1, \dots, x_k) \in \mathrm{Gr}_d(\mathfrak{r}) \times \mathfrak{r}^k \mid u \ni x_1, \dots, u \ni x_k\}$$

over $\mathrm{Gr}_d(\mathfrak{r})$ so that $\mathcal{E}^{(k)}$ is vector bundle of rank kd over X_R. In particular, $\mathcal{E}^{(1)} = \mathcal{E}$. According to [Hir64], there exists a desingulartization Γ of X_R with morphism ρ such that the restriction of ρ to $\rho^{-1}(X_{R\rm sm})$ is an isomorphism onto $X_{R\rm sm}$. Let $\widetilde{\mathcal{E}^{(1)}}$ be the following fiber product

$$\begin{tikzcd} \widetilde{\mathcal{E}^{(1)}} \arrow[r, "\overline{\rho}"] \arrow[d] & \mathcal{E}^{(1)} \arrow[d, "\varpi_1"] \\ \Gamma \arrow[r, "\rho"'] & X_R \end{tikzcd}$$

with $\overline{\rho}$ the restriction map. Then $\widetilde{\mathcal{E}^{(1)}}$ is a vector bundle of rank d over Γ. In particular, it is a smooth variety since Γ is smooth.

Let O be a trivialization open subset of the vector bundle $\mathcal{E}^{(1)}$ and let Φ_1 be a local trivialization over O of $\mathcal{E}^{(1)}$, whence the following commutative diagram

$$\begin{tikzcd} \varpi_1^{-1}(O) \arrow[r, "\Phi_1"] \arrow[dr, "\varpi_1"'] & O \times \Bbbk^d \arrow[d, "\mathrm{pr}_1"] \\ & O \end{tikzcd} .$$

Then O is a trivialization open subset of the vector bundle $\mathcal{E}^{(k)}$. The variety $\mathcal{E}^{(1)}$ is a closed subbundle of $\mathcal{E}^{(k)}$ over X_R and for some local trivialization Φ over O of $\mathcal{E}^{(k)}$, we have the following commutative diagram

$$\begin{tikzcd} \varpi_k^{-1}(O) \arrow[r, "\Phi"] \arrow[dr, "\varpi_k"'] & O \times \Bbbk^{kd} \arrow[d, "\mathrm{pr}_1"] \\ & O \end{tikzcd} ,$$

Φ_1 is the restriction of Φ to $\varpi_1^{-1}(O)$ and $\Phi(\varpi_1^{-1}(O)) = O \times \Bbbk^d \times \{0\}$.

Lemma 5.8. *Suppose $k \geqslant 2$. Denote by μ a generator of $\Omega_{\Bbbk^{kd}}$ and by $\tilde{\rho}$ the restriction of $\rho \times \mathrm{id}_{\Bbbk^{kd}}$ to $\rho^{-1}(O) \times \Bbbk^{kd}$.*

(i) *The sheaf $\Omega_{\mathcal{E}^{(k)}_{\rm sm}}$ has a global section ω without zero.*

(ii) *The sheaf $\Omega_{O_{\rm sm}}$ has a global section ω_O without zero.*

(iii) *For some p in $\Bbbk[O \times \Bbbk^{kd}] \setminus \{0\}$, $\tilde{\rho}^*(p(\omega_O \wedge \mu))$ has a regular extension to $\rho^{-1}(O) \times \Bbbk^{kd}$.*

Proof. (i) According to Proposition 5.6 and Lemma 5.7(iii), $\Omega_{\tau_k^{-1}(V^{(k)})}$ has a global section without zero. By Lemma 5.7(ii), $\tau_k^{-1}(V^{(k)})$ is a smooth big open subset of $\mathcal{E}^{(k)}$. So, by Lemma A.1, $\Omega_{\mathcal{E}^{(k)}_{\rm sm}}$ has a global section without zero.

(ii) Since μ is a generator of $\Omega_{\Bbbk^{kd}}$, there exists a unique ν in $\Bbbk[\Bbbk^{kd}]\otimes_{\Bbbk}\Gamma(O_{\rm sm},\Omega_{O_{\rm sm}})$ such that

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(O_{\rm sm})}) = \nu\wedge\mu.$$

Moreover, ν has no zero since so has ω. Let V be an affine open subset of $O_{\rm sm}$ such that the restriction of $\Omega_{O_{\rm sm}}$ to V is locally free, generated by the local section ω_V. Then for some p_V in $\Bbbk[V\times\Bbbk^{kd}]$,

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(V)}) = p_V\omega_V\wedge\mu. \tag{4}$$

Then p_V has no zero since so has $\nu\wedge\mu$. As a result, p_V is in $\Bbbk[V]$ and $p_V\omega_V$ is a local section of $\Omega_{O_{\rm sm}}$ without zero. By the unicity of the decomposition (4), for two different affine open subsets V and V' as above, the differential forms $p_V\omega_V$ and $p_{V'}\omega_{V'}$ have the same restriction to $V\cap V'$. As a result, since such affine open subsets cover $O_{\rm sm}$, for some global section ω_O of $\Omega_{O_{\rm sm}}$,

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(O_{\rm sm})}) = \omega_O\wedge\mu.$$

Moreover, ω_O is unique and has no zero.

(iii) Let ω_1 be a generator of $\Omega_{\rm r}$ and let μ_1 be a generator of $\Omega_{\Bbbk^d}$. By (i), $\omega_O\wedge\mu_1$ is a global section of $\Omega_{O_{\rm sm}\times\Bbbk^d}$, without zero. So for some regular function p on $O_{\rm sm}\times\Bbbk^d$,

$$\Phi_{1*}((\tau_1)^*(\omega_1)\,|_{\varpi_1^{-1}(O_{\rm sm})}) = p\omega_O\wedge\mu_1. \tag{5}$$

According to Theorem 4.11, X_R is normal. Then so is O and p has a regular extension to $O\times\Bbbk^d$. Denote again by p this extension. According to Equality (5), the differential form $\tilde\rho^*(p\omega_O\wedge\mu_1)$ on $\rho^{-1}(O_{\rm sm})\times\Bbbk^d$ has a regular extension to $\rho^{-1}(O)\times\Bbbk^d$. In fact, denoting by $\overline{\Phi_1}$ the local trivialization over $\rho^{-1}(O)$ of $\widetilde{\mathcal{E}^{(1)}}$ such that the following diagram

$$\begin{tikzcd}
(\varpi_1\circ\overline{\rho}^{-1})(O) \arrow[r, "\overline{\Phi_1}"] \arrow[d, "\overline{\rho}"'] & \rho^{-1}(O)\times\Bbbk^d \arrow[d, "\tilde\rho"] \\
\varpi_1^{-1}(O) \arrow[r, "\Phi_1"'] & O\times\Bbbk^d
\end{tikzcd}$$

is commutative, it is the restriction to $\rho^{-1}(O_{\rm sm})\times\Bbbk^d$ of

$$\overline{\Phi_1}_*((\tau_1\circ\overline{\rho})^*(\omega_1)\,|_{(\varpi_1\circ\overline{\rho}^{-1})^{-1}(O)}).$$

For some generator μ' of $\Omega_{\Bbbk^{(k-1)d}}$, $\mu=\mu_1\wedge\mu'$ and $\Bbbk[O\times\Bbbk^d]$ is naturally embedded in $\Bbbk[O\times\Bbbk^{kd}]$. As a result, $\tilde\rho^*(p\omega_O\wedge\mu)$ has a regular extension to $\rho^{-1}(O)\times\Bbbk^{kd}$. □

Proposition 5.9. *The variety X_R is Gorenstein with rational singularities.*

Proof. According to Theorem 4.11, X_R is normal and Cohen-Macaulay. Then by Lemma 5.8,(ii) and (iii) and Corollary A.5, $O\times\Bbbk^{kd}$ is Gorenstein with rational singularities. Then so is O by Lemma B.1,(i) and (ii). Since there is a cover of X_R by open subsets as O, X_R is Gorenstein with rational singularities. □

As already mentioned, $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{h},*}$, whence Theorem 1.1 by Proposition 5.9.

Let X be an affine irreducible normal variety.

Lemma A.1. *Let Y be a smooth big open subset of X.*

(i) *All regular differential form of top degree on Y has a unique regular extension to $X_{\rm sm}$.*

(ii) *Suppose that ω is a regular differential form of top degree on Y, without zero. Then the regular extension of ω to $X_{\rm sm}$ has no zero.*

Proof. (i) Since $\Omega_{X_{\rm sm}}$ is a locally free module of rank one, there is an affine open cover $O_1,\ldots,O_k$ of $X_{\rm sm}$ such that the restriction of $\Omega_{X_{\rm sm}}$ to O_i is a free $\mathcal{O}_{O_i}$-module generated by some section ω_i. For $i=1,\ldots,k$, set $O'_i := O_i\cap Y$. Let ω be a regular differential form of top degree on Y. For $i=1,\ldots,k$, for some regular function a_i on O'_i, $a_i\omega_i$ is the restriction of ω to O'_i. As Y is a big open subset of X, O'_i is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1\leqslant i,j\leqslant k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O'_i\cap O'_j$ and $O_i\cap O_j$ since $\Omega_{X_{\rm sm}}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{X_{\rm sm}}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to $X_{\rm sm}$ and this extension is unique since Y is dense in $X_{\rm sm}$ and $\Omega_{X_{\rm sm}}$ is torsion free.

(ii) Suppose that ω has no zero. Let Σ be the nullvariety of ω' in $X_{\rm sm}$. If it is not empty, Σ has codimension 1 in $X_{\rm sm}$. As Y is a big open subset of X, $\Sigma\cap X_{\rm sm}$ is not empty if so is Σ. As a result, Σ is empty. □

Denote by ι the canonical injection from $X_{\rm sm}$ into X.

Lemma A.2. *Suppose that $\Omega_{X_{\rm sm}}$ has a global section ω without zero. Then the $\mathcal{O}_X$-module $\iota_*(\Omega_{X_{\rm sm}})$ is free of rank* 1. *More precisely, the morphism θ:*

$$\mathcal{O}_X \xrightarrow{\ \theta\ } \iota_*(\Omega_{X_{\rm sm}})\ , \qquad \psi\longmapsto \psi\omega$$

is an isomorphism.

Proof. For φ a local section of $\iota_*(\Omega_{X_{\rm sm}})$ above the open subset U of X, for some regular function ψ on $U\cap X_{\rm sm}$, $\psi\omega$ is the restriction of φ to $U\cap X_{\rm sm}$. Since X is normal, so is U and $U\cap X_{\rm sm}$ is a big open subset of U. Hence ψ has a regular extension to U. As a result, there exists a well defined morphism from $\iota_*(\Omega_{X_{\rm sm}})$ to $\mathcal{O}_X$ whose inverse is θ. □

According to [Hir64], X has a desingularization Z with morphism τ such that the restriction of τ to $\tau^{-1}(X_{\rm sm})$ is an isomorphism onto $X_{\rm sm}$. For U open subset of X, denote by τ_U the restriction of τ to $\tau^{-1}(U)$.

Proposition A.3. *Suppose that X is Cohen-Macaulay and that there exists a morphism $\mu:\ \mathcal{O}_Z\longrightarrow\Omega_Z$ such that for some p in $\Bbbk[X]$, $\tau_*\mu$ is an isomorphism onto $p\tau_*(\Omega_Z)$. Then X has rational singularities.*

The following proof is the weak variation of the proof of [Hi91, Lemma 2.3].

Proof. Since Z and X are varieties over $\Bbbk$, we have the commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "\tau"] \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd}$$

According to [H66, V. §10.2], $p^!(\Bbbk)$ and $q^!(\Bbbk)$ are dualizing complexes over Z and X respectively. Furthermore, by [H66, VII, 3.4] or [Hi91, 4.3,(ii)], $p^!(\Bbbk)[-\dim Z]$ equals Ω_Z. Set $\mathcal{D} := q^!(\Bbbk)[-\dim Z]$ so that $\tau^!(\mathcal{D})=\Omega_Z$ by [H66, VII, 3.4] or [Hi91, 4.3,(iv)]. In particular, $\mathcal{D}$ is dualizing over X.

Since τ is a projective morphism, we have the isomorphism

$$\mathrm{R}\tau_*(\mathrm{R}\mathscr{H}om_Z(\Omega_Z,\Omega_Z)) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\Omega_Z),\mathcal{D}) \tag{6}$$

by [H66, VII, 3.4] or [Hi91, 4.3,(iii)]. Since $\mathrm{H}^i(\mathrm{R}\mathscr{H}om_Z(\Omega_Z,\Omega_Z)) = \mathcal{O}_Z$ for $i = 0$ and 0 for $i > 0$, the left hand side of (6) can be identified with $\mathrm{R}\tau_*(\mathcal{O}_Z)$.

According to Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}\tau_*(\Omega_Z)$ has only cohomology in degree 0. Since τ is projective and birational and Z is normal, $\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$. So by assumption of the proposition,

$$\mathrm{R}\tau_*(\Omega_Z) \approx \frac{1}{p}\mathcal{O}_X,$$

whence

$$\mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\Omega_Z),\mathcal{D}) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$$

and (6) can be rewritten as

$$\mathrm{R}\tau_*(\mathcal{O}_Z) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}. \tag{7}$$

Since X is Cohen-Macaulay, $\mathcal{D}$ has cohomology in only one degree. So, by flatness of the $\mathcal{O}_X$-module $p\mathcal{O}_X$, $p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$ has cohomology in only one degree. As a result, by (7), $\mathrm{R}^i\tau_*(\mathcal{O}_Z) = 0$ for $i > 0$, that is X has rational singularities. □

Denote by $\mathcal{M}$ the cohomology in degree 0 of $\mathcal{D}$.

Lemma A.4. *Suppose that X has rational singularities. Then the $\mathcal{O}_X$-modules $\tau_*(\Omega_Z)$ and $\mathcal{M}$ are isomorphic. In particular, $\tau_*(\Omega_Z)$ has finite injective dimension.*

Proof. Since X has rational singularities, $\mathrm{R}\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$ and $\mathcal{D}$ has only cohomology in degree 0. Moreover, by Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}\tau_*(\Omega_Z)$ has only cohomology in degree 0, whence $R\tau_*(\Omega_Z) = \tau_*(\Omega_Z)$. Then, by (6), we have the isomorphism

$$\mathcal{O}_X \longrightarrow \mathscr{H}om_X((\tau)_*(\Omega_Z),\mathcal{M}) .$$

As $\mathcal{D}$ is dualizing, we have the isomorphism

$$R\tau_*(\Omega_Z) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(R\tau_*(\Omega_Z),\mathcal{D}),\mathcal{D})$$

whence the isomorphism $\tau_*(\Omega_Z) \longrightarrow \mathcal{M}$ by (6). As a result, $\tau_*(\Omega_Z)$ has finite injective dimension since so has $\mathcal{M}$. □

Corollary A.5. *Let Y be a smooth big open subset of X. Suppose that the following conditions are verified:*

1. *X is Cohen-Macaulay,*
2. *Ω_Y has a global section ω without zero,*
3. *for some global section ω_Z of Ω_Z and for some p in $\Bbbk[X]\setminus\{0\}$, the restriction of ω_Z to $\tau^{-1}(Y)$ is equal to $p\tau_Y^*(\omega)$.*

Then X is Gorenstein with rational singularities. Moreover, its canonical module is free of rank 1.

Proof. According to Lemma A.1(ii), ω has a unique regular extension to X_{sm} and this extension has no zero. Denote again by ω this extension. Since Z is irreducible, $\tau^{-1}(Y)$ is dense in $\tau^{-1}(X_{\mathrm{sm}})$ so that the restriction of ω_Z to $\tau^{-1}(X_{\mathrm{sm}})$ is equal to $p\tau_{X_{\mathrm{sm}}}^*(\omega)$ since Ω_Z has no torsion. Denote by μ the morphism

$$\mathcal{O}_Z \xrightarrow{\mu} \Omega_Z , \qquad \varphi \longmapsto \varphi\omega_Z.$$

Let U be an open subset of X and ν a local section of $\tau_*(\Omega_Z)$ above U. Since ω has no zero and $\tau_{U_{\rm sm}}$ is an isomorphism onto $U_{\rm sm}$,

$$\nu|_{\tau^{-1}(U_{\rm sm})} = \tau^*_{U_{\rm sm}}(\varphi\omega|_{U_{\rm sm}})$$

for some φ in $\Bbbk[U_{\rm sm}]$, whence

$$p\nu|_{\tau^{-1}(U_{\rm sm})} = \varphi\circ\tau_{U_{\rm sm}}(\omega_Z|_{\tau^{-1}(U_{\rm sm})})$$

by Condition (3). Since X is normal, so is U and $U_{\rm sm}$ is a big open subset of U. Hence φ has a regular extension to U. Denoting again by φ this extension,

$$p\nu = \varphi\circ\tau_U(\omega_Z|_{\tau^{-1}(U)})$$

since Z is irreducible and Ω_Z has no torsion. As a result the morphism

$$\tau_*\mu : \ \tau_*(\mathcal{O}_Z) \longrightarrow p\tau_*(\Omega_Z)$$

is an isomorphism since it is obviously injective. So, by Proposition A.3, X has rational singularities. In particular, by [KK73, p.50], $\tau_*(\Omega_X) = \iota_*(\Omega_X)$. Then, by Lemma A.2, the canonical module of X is free of rank 1 and by Lemma A.4, X is Gorenstein. □

Appendix B. About singularities

In this section we recall a well known result. Let X be a variety and Y a vector bundle over X. Denote by τ the bundle projection.

Lemma B.1. (i) *If Y is Gorenstein, then X is Gorenstein.*

(ii) *The variety X has rational singularities if and only if so has Y.*

(iii) *If X is Cohen-Macaualay, then so is Y.*

Proof. Let y be in Y, $x := \tau(y)$. Denote by $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,y}}$ the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively.

(i) Since Y is a vector bundle over X, $\widehat{\mathcal{O}_{Y,y}}$ is a ring of formal series over $\widehat{\mathcal{O}_{X,x}}$. By [Bru, Proposition 3.1.19,(c)], $\widehat{\mathcal{O}_{Y,y}}$ is Gorenstein. So, by [Bru, Proposition 3.1.19,(b)], $\widehat{\mathcal{O}_{X,x}}$ is Gorenstein. Then by [Bru, Proposition 3.1.19,(c)], $\mathcal{O}_{X,x}$ is Gorenstein, whence the assertion.

(ii) Since Y is a vector bundle over X, then there exists a cover of X by open subsets O, such that $\tau^{-1}(O)$ is isomorphic to $O\times\Bbbk^m$ with $m = \dim Y - \dim X$. According to [KK73, p.50], $O\times\Bbbk^m$ has rational singularities if and only if so has O, whence the assertion since a variety has rational singularities if and only it has a cover by open subsets having rational singularities.

(iii) According to [MA86, Ch. 6, Theorem 17.7], a polynomial algebra over a Cohen-Macaulay algebra is Cohen-Macaulay. Hence for O open subset of X as in (ii), $\tau^{-1}(O)$ is Cohen-Macaulay, whence the assertion since there is a cover of Y by open subsets as $\tau^{-1}(O)$. □

References

[Bol91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicae, **24** (1991), **n°1**, p. 253–274.

[Bou98] N. Bourbaki, *Algèbre commutative, Chapitre 10, Éléments de mathématiques*, Masson (1998), Paris.

[Boutot87] J-François. Boutot, *Singularités rationnelles et quotients par les groupes réductifs*, Inventiones Mathematicae **88** (1987), p. 65–68.

[Bru] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **n°39**, Cambridge University Press, Cambridge (1996).

[CZ16] J-Y. Charbonnel and M. Zaiter *On the Commuting variety of a reductive Lie algebra and other related varieties*, Journal of Algebra **458** (2016), pp. 445–497.

[El78] R. Elkik, *Singularités rationnelles et déformations*, Inventiones Mathematicae **47** (1978), p. 139–147.

[El81] R. Elkik, *Rationalité des singularités canoniques*, Inventiones Mathematicae **64** (1981), p. 1–6.

[EGAII] A. Grothendieck *Éléments de Géométrie Algébrique II - Étude globale élémentaire de quelques classes de morphismes.* Publications Mathématiques **n°8** (1961), Le Bois-Marie, Bures sur Yvette.

[EGAIV] A. Grothendieck *Éléments de Géométrie Algébrique IV -Étude locale des schémas et des morphismes de schémas.* Publications Mathématiques **n°28–32** (1966-67), Le Bois-Marie, Bures sur Yvette.

[Gi12] V. Ginzburg *Isospectral commuting variety, the Harish-Chandra $\mathcal{D}$-module, and principal nilpotent pairs* Duke Mathematical Journal, **161** (2012), p. 2023–2111.

[GR70] H. Grauert, O. Riemenschneider *Verschwindungssätze für analytiscbe Kohomologie-gruppen auf komplexen Räumen* Inventiones Mathematicae **11** (1970), p. 263–292.

[H66] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics **n°20** (1966), Springer-Verlag, Berlin Heidelberg New York.

[H77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **n°52** (1977), Springer-Verlag, Berlin Heidelberg New York.

[Ha99] M. Haiman, *Macdonald polynomials and geometry. New perspectives in algebraic combinatorics (Berkeley, CA,1996–97)*, Math. Sci. Res. Inst. Publ. **38** (1999), Cambridge Univ. Press, Cambridge, p. 207–254.

[Ha02] M. Haiman, *Combinatorics, symmetric functions and Hilbert schemes*, Current Developments in Mathematics, **1** (2002), p. 39–111.

[Hi91] V. Hinich, *On the singularities of nilpotent orbits*, Israel Journal of Mathematice, **73** (1991), p. 297–308.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I,II*, Annals of Mathematics **79** (1964), p. 109–203 and p. 205–326.

[Hu95] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, **n°43**, (1995).

[Ke77] G. R. Kempf, *Some quotient varieties have rational singularities*, Michigan Mathematical Journal **24** (1977), p. 347–352.

[KK73] G. R. Kempf, F. Knusdon, D. Mumford and B. Saint-Donat, *Toroidal embeddings*, Lecture Notes in Mathematics **n°339** (1973), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Ko63] B. Kostant, *Lie group representations on polynomial rings*, American Journal of Mathematics **85** (1963), p. 327–404.

[MA86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics **n°8** (1986), Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.

[Mebk89] Z. Mebkhout, Systèmes différentiels, Le formalisme des six opérations de Grothendieck pour les D_X-modules cohérents, Travaux en cours, **35**, Hermann, Paris.

[MF78] A.S. Mishchenko and A.T. Fomenko, *Euler equations on Lie groups*, Math. USSR-Izv. **12** (1978), p. 371–389.

[Mu65] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und Ihrer Grenzgzbiete **band 34** (1965), Springer-Verlag, Berlin, Heidelberg, New York.

[Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **n°1358** (1988), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Po08] V.L. Popov *Irregular and singular loci of commuting varieties*, Transformation Groups **13** (2008), p. 819–837.

[Po08] V.L. Popov and E. B. Vinberg, *Invariant Theory, in: Algebraic Geometry IV*, Encyclopaedia of MathematicalSciences **n°55** (1994), Springer-Verlag, Berlin, p.123–284.

[Ri79] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Mathematica **38** (1979), p. 311–322.

[Sh94] I.R. Shafarevich, *Basic algebraic geometry 2*, Springer-Verlag (1994), Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong-Kong, Barcelona, Budapest.

[TY05] P. Tauvel and R.W.T. Yu, *Lie algebras and algebraic groups*, Monographs in Mathematics (2005), Springer, Berlin Heidelberg New York.

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